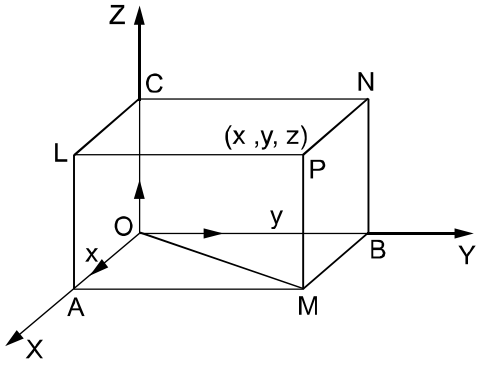
26

Three-Dimensional Geometry

COORDINATES OF A POINT IN SPACE Let O be the origin, and let OX, OY and OZ be three mutually perpendicular lines, taken as the x-axis, the y-axis and the z-axis respectively in such a way that they form a right-handed system.

The planes, YOZ, ZOX and XOY are respectively known as *yz-plane*, *zx-plane* and *xy-plane*.

These planes, known as coordinate planes, divide the space into eight parts, called *octants*.



Let P be a point in space. Through P, draw planes parallel to coordinate planes, and meeting the axes OX, OY and OZ in points A, B and C respectively. Complete the parallelepiped whose coterminous edges are OA, OB and OC.

Let $OA = x$, $OB = y$ and $OC = z$.

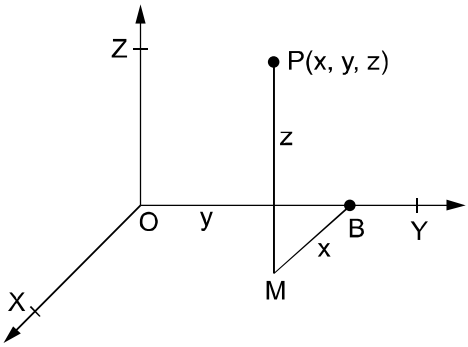
Then, we say that the *coordinates of P are* (x, y, z).

REMARK In fact, we have,

(i) x = distance of P from yz-plane;

(ii) y = distance of P from xz-plane;

(iii) z = distance of P from xy-plane.



As a consequence of the above facts, we find that

(i) Every point in yz-plane has x-coordinate zero.

(ii) Every point in xz-plane has y-coordinate zero.

(iii) Every point in xy-plane has z-coordinate zero.

(iv) Any point on the x-axis is of the form $(x, 0, 0)$.

(v) Any point on the y-axis is of the form $(0, y, 0)$.

(vi) Any point on the z-axis is of the form $(0, 0, z)$.

The following table shows the signs of coordinates in eight octants.

	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

EXAMPLE 1 *In which octant does the given point lie?*

(i) $(-2, 4, 3)$ *(ii)* $(3, -2, -5)$ *(iii)* $(-6, 3, -4)$

(iv) $(-3, -1, 4)$ *(v)* $(1, -3, 6)$ *(vi)* $(4, 7, -2)$

SOLUTION It is clear from the above table that the point

(i) $(-2, 4, 3)$ lies in octant II;

(ii) $(3, -2, -5)$ lies in octant VIII;

(iii) $(-6, 3, -4)$ lies in octant VI;

(iv) $(-3, -1, 4)$ lies in octant III;

(v) $(1, -3, 6)$ lies in octant IV;

(vi) $(4, 7, -2)$ lies in octant V.

EXAMPLE 2 *If a point lies on the y-axis then what are its x-coordinate and z-coordinate?*

SOLUTION If a point lies on the y-axis then its x-coordinate is 0 and its z-coordinate is 0.

EXAMPLE 3 *If a point lies in xy-plane then what is its z-coordinate?*

SOLUTION If a point lies in xy-plane then its z-coordinate is 0.

EXAMPLE 4 *In which plane does the point $(0, 5, -4)$ lie?*

SOLUTION Clearly, the point $(0, 5, -4)$ lies in the yz-plane.

EXERCISE 26A

1. If a point lies on the z-axis then find its x-coordinate and y-coordinate.

2. If a point lies on yz-plane then what is its x-coordinate?

3. In which plane does the point $(4, -3, 0)$ lie?

4. In which octant does each of the given points lie?

(i) $(-4, -1, -6)$ (ii) $(2, 3, -4)$ (iii) $(-6, 5, -1)$

(iv) $(4, -3, -2)$ (v) $(-1, -6, 5)$ (vi) $(4, 6, 8)$

ANSWERS (EXERCISE 26A)

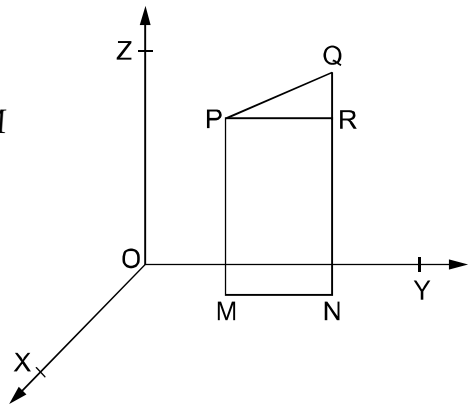
1. $x = 0, y = 0$ **2.** $x = 0$ **3.** xy-plane

4. (i) VII (ii) V (iii) VI (iv) VIII (v) III (vi) I

THEOREM 1 **(Distance Formula)** *Prove that the distance between the points* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *is given by*

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

PROOF Let O be the origin, and let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the given point. From P and Q, draw perpendiculars PM and QN respectively on the xy-plane.



Then, the coordinates of M and N are

$$M(x_1, y_1, 0) \text{ and } N(x_2, y_2, 0).$$

The two-dimensional coordinates of M and N referred to OX and OY are

$$M(x_1, y_1) \text{ and } N(x_2 y_2).$$

$$\therefore \quad MN^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Now, from P draw $PR \perp QN$.

Then, PR is parallel and equal to MN.

Now, in right triangle PRQ, we have

$$\begin{aligned} PQ^2 &= PR^2 + RQ^2 \\ &= MN^2 + (QN - RN)^2 \\ &= MN^2 + (QN - PM)^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad [\because\ PM = z_1 \text{ and } QN = z_2]. \end{aligned}$$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

COROLLARY *The distance of the point* $P(x, y, z)$ *from the origin* $O(0, 0, 0)$ *is*

$$OP = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} = \sqrt{x^2 + y^2 + z^2}.$$

EXAMPLE 1 *Find the distance between the points* $A(-2, 1, -3)$ *and* $B(4, 3, -6)$.

SOLUTION We have

$$AB = \sqrt{\{[4 - (-2)]^2 + (3 - 1)^2 + [-6 - (-3)]^2\}}$$

$$= \sqrt{\{6^2 + 2^2 + (-3)^2\}} = \sqrt{49} = 7 \text{ units}.$$

EXAMPLE 2 *Show that the points* $A(0, 7, 10)$, $B(-1, 6, 6)$ *and* $C(-4, 9, 6)$ *form an isosceles right-angled triangle.*

SOLUTION We have

$$AB = \sqrt{(-1 - 0)^2 + (6 - 7)^2 + (6 - 10)^2} = \sqrt{18} = 3\sqrt{2},$$

$$BC = \sqrt{(-4 + 1)^2 + (9 - 6)^2 + (6 - 6)^2} = \sqrt{18} = 3\sqrt{2},$$

and $AC = \sqrt{(-4 - 0)^2 + (9 - 7)^2 + (6 - 7)^2} = \sqrt{36} = 6.$

Clearly, $AB = BC$ and $AB^2 + BC^2 = (18 + 18) = 36 = AC^2$.

Hence, triangle ABC is an isosceles right-angled triangle.

EXAMPLE 3 *Prove that the points* $A(3, -2, 4)$, $B(1, 1, 1)$ *and* $C(-1, 4, -2)$ *are collinear.*

SOLUTION We have

$$AB = \sqrt{(1-3)^2 + (1+2)^2 + (1-4)^2} = \sqrt{4+9+9} = \sqrt{22},$$

$$BC = \sqrt{(-1-1)^2 + (4-1)^2 + (-2-1)^2} = \sqrt{4+9+9} = \sqrt{22},$$

and $AC = \sqrt{(-1-3)^2 + (4+2)^2 + (-2-4)^2} = \sqrt{16+36+36}$

$$= \sqrt{88} = 2\sqrt{22}.$$

$\therefore \quad AB + BC = AC.$

This shows that the given points are collinear.

EXAMPLE 4 *Find the equation of the curve formed by the set of all points whose distances from the points* $(3, 4, -5)$ *and* $(-2, 1, 4)$ *are equal.*

SOLUTION Let $P(x, y, z)$ be any point on the given curve, and let $A(3, 4, -5)$ and $B(-2, 1, 4)$ be the given points.

Then, $PA = PB$

$\Rightarrow \quad PA^2 = PB^2$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 + (z+5)^2 = (x+2)^2 + (y-1)^2 + (z-4)^2$

$\Rightarrow \quad 10x + 6y - 18z - 29 = 0.$

Hence, the required curve is $10x + 6y - 18z - 29 = 0$.

EXAMPLE 5 *Find the equation of the curve formed by the set of all those points the sum of whose distances from the points* $A(4, 0, 0)$ *and* $B(-4, 0, 0)$ *is* 10 *units.*

SOLUTION Let $P(x, y, z)$ be an arbitrary point on the given curve. Then,

$$PA + PB = 10$$

$$\Rightarrow \quad \sqrt{(x-4)^2 + y^2 + z^2} + \sqrt{(x+4)^2 + y^2 + z^2} = 10$$

$$\Rightarrow \quad \sqrt{(x+4)^2 + y^2 + z^2} = 10 - \sqrt{(x-4)^2 + y^2 + z^2} \qquad \text{... (i)}$$

$$\Rightarrow \quad (x+4)^2 + y^2 + z^2 = 100 + (x-4)^2 + y^2 + z^2 - 20\sqrt{(x-4)^2 + y^2 + z^2}$$

[on squaring both sides of (i)]

$$\Rightarrow \quad 16x = 100 - 20\sqrt{(x-4)^2 + y^2 + z^2}$$

$$\Rightarrow \quad 5\sqrt{(x-4)^2 + y^2 + z^2} = (25 - 4x)$$

$$\Rightarrow \quad 25[(x-4)^2 + y^2 + z^2] = 625 + 16x^2 - 200x$$

$$\Rightarrow \quad 9x^2 + 25y^2 + 25z^2 - 225 = 0.$$

Hence, the required equation of the curve is

$$9x^2 + 25y^2 + 25z^2 - 225 = 0.$$

SOME USEFUL RESULTS

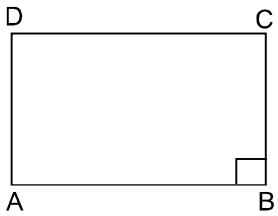
A quadrilateral $ABCD$ is a

(i) parallelogram, if $AB = CD$ and $BC = DA$;

(ii) rectangle, if $AB = CD$, $BC = DA$ and $AC = BD$;

(iii) rhombus, if $AB = BC = CD = DA$ but $AC \neq BD$;

(iv) square, if $AB = BC = CD = DA$ and $AC = BD$.



EXERCISE 26B

1. Find the distance between the points:

(i) $A(5, 1, 2)$ and $B(4, 6, -1)$ (ii) $P(1, -1, 3)$ and $Q(2, 3, -5)$

(iii) $R(1, -3, 4)$ and $S(4, -2, -3)$ (iv) $C(9, -12, -8)$ and the origin

2. Show that the points $A(1, -1, -5)$, $B(3, 1, 3)$ and $C(9, 1, -3)$ are the vertices of an equilateral triangle.

3. Show that the points $A(4, 6, -5)$, $B(0, 2, 3)$ and $C(-4, -4, -1)$ form the vertices of an isosceles triangle.

4. Show that the points $A(0, 1, 2)$, $B(2, -1, 3)$ and $C(1, -3, 1)$ are the vertices of an isosceles right-angled triangle.

5. Show that the points $A(1, 1, 1)$, $B(-2, 4, 1)$, $C(1, -5, 5)$ and $D(2, 2, 5)$ are the vertices of a square.

6. Show that the points $A(1, 2, 3)$, $B(-1, -2, -1)$, $C(2, 3, 2)$ and $D(4, 7, 6)$ are the vertices of a parallelogram. Show that $ABCD$ is not a rectangle.

7. Show that the points $P(2, 3, 5)$, $Q(-4, 7, -7)$, $R(-2, 1, -10)$ and $S(4, -3, 2)$ are the vertices of a rectangle.

8. Show that the points $P(1, 3, 4)$, $Q(-1, 6, 10)$, $R(-7, 4, 7)$ and $S(-5, 1, 1)$ are the vertices of a rhombus.

9. Show that $D(-1, 4, -3)$ is the circumcentre of triangle ABC with vertices $A(3, 2, -5)$, $B(-3, 8, -5)$ and $C(-3, 2, 1)$.

10. Show that the following points are collinear:

(i) $A(-2, 3, 5)$, $B(1, 2, 3)$ and $C(7, 0, -1)$

(ii) $A(3, -5, 1)$, $B(-1, 0, 8)$ and $C(7, -10, -6)$

(iii) $P(3, -2, 4)$, $Q(1, 1, 1)$ and $R(-1, 4, 2)$

11. Find the equation of the curve formed by the set of all points which are equidistant from the points $A(-1, 2, 3)$ and $B(3, 2, 1)$.

12. Find the point on the y-axis which is equidistant from the points $A(3, 1, 2)$ and $B(5, 5, 2)$.

13. Find the point on the z-axis which is equidistant from the points $A(1, 5, 7)$ and $B(5, 1, -4)$.

14. Find the coordinates of the point which is equidistant from the points $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$ and $O(0, 0, 0)$.

15. Find the point in yz-plane which is equidistant from the points $A(3, 2, -1)$, $B(1, -1, 0)$ and $C(2, 1, 2)$.

16. Find the point in xy-plane which is equidistant from the points $A(2, 0, 3)$, $B(0, 3, 2)$ and $C(0, 0, 1)$.

ANSWERS (EXERCISE 26B)

1. (i) $\sqrt{35}$ units (ii) 9 units (iii) $\sqrt{59}$ units (iv) 17 units

11. $2x - z = 0$ **12.** $(0, 3, 0)$ **13.** $\left(0, 0, \frac{3}{2}\right)$ **14.** $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$

15. $\left(0, \frac{31}{16}, \frac{-3}{16}\right)$ **16.** $(3, 2, 0)$

HINTS TO SOME SELECTED QUESTIONS

6. Show that $AB = CD$, $BC = DA$ and $AC \neq BD$.

7. Show that $AB = CD$, $BC = DA$ and $AC = BD$.

8. Show that $PQ = QR = RS = SP$ and $PR \neq QS$.

9. Show that $DA = DB = DC$.

11. Let $P(x, y, z)$ be an arbitrary point on the given curve. Then,

$PA = PB \Rightarrow PA^2 = PB^2$.

12. Any point on the y-axis is of the form $P(0, y, 0)$.

14. Let the required point be $P(x, y, z)$. Then,

$PA^2 = PB^2 = PC^2 = PO^2 \Leftrightarrow PA^2 - PO^2 = 0, PB^2 - PO^2 = 0$ and $PC^2 - PO^2 = 0$.

15. Any point in the yz-plane is of the form $P(0, y, z)$.

$|AP| = |BP| = |CP| \Leftrightarrow AP^2 = BP^2$ and $BP^2 = CP^2$.

$(0-3)^2 + (y-2)^2 + (z+1)^2 = (0-1)^2 + (y+1)^2 + (z-0)^2$

and $(0-1)^2 + (y+1)^2 + (z-0)^2 = (0-2)^2 + (y-1)^2 + (z-2)^2$

$\Rightarrow \quad 3y - z - 6 = 0 \quad$ and $\quad 4y + 4z - 7 = 0$.

On solving, we get $y = \frac{31}{16}$ and $z = \frac{-3}{16}$.

Hence, the required point is $\left(0, \frac{31}{16}, \frac{-3}{16}\right)$.

SECTION FORMULAE

THEOREM 2 **(Section Formula)** *Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let $R(x, y, z)$ be a point on PQ dividing it in the ratio $m : n$. Prove that*

$$x = \frac{mx_2 + nx_1}{m+n},\ y = \frac{my_2 + ny_1}{m+n} \text{ and } z = \frac{mz_2 + nz_1}{m+n}.$$

PROOF From P, Q and R, draw perpendiculars PL, QM and RN on the xy-plane. Also, draw $PS \perp QM$, meeting QM and RN at S and T respectively.

From similar triangles PRT and PQS, we have

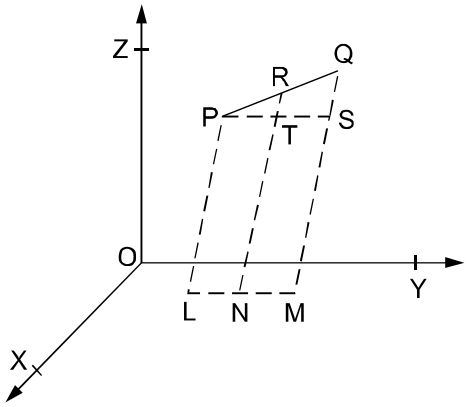
$$\frac{RT}{QS} = \frac{PR}{PQ}$$

$$\Rightarrow \frac{RN - TN}{QM - SM} = \frac{m}{m+n}$$

$$\Rightarrow \frac{RN - PL}{QM - PL} = \frac{m}{m+n}$$

$$\Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{m}{m+n}$$

$$\Rightarrow z = \frac{mz_2 + nz_1}{m+n}.$$



Similarly, $x = \frac{mx_2 + nx_1}{m+n}$ and $y = \frac{my_2 + ny_1}{m+n}$.

Hence, $x = \frac{mx_2 + nx_1}{m+n},\ y = \frac{my_2 + ny_1}{m+n},\ z = \frac{mz_2 + nz_1}{m+n}$.

COROLLARY 1 *The coordinates of the midpoint of the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

PROOF Let R be the midpoint of PQ.

Putting $m = n = 1$ in the above result, we find that the coordinates of R are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

COROLLARY 2 *The coordinates of a point R which divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio $m : n$ are*

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right).$$

PROOF Replacing n by $-n$ in Theorem 2, we find that the coordinates of R are

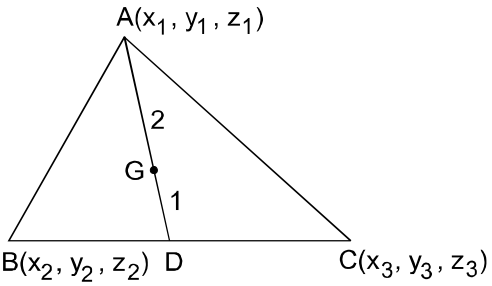
$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right).$$

CENTROID OF A TRIANGLE

THEOREM 3 *Show that the centroid of the triangle with vertices* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ *and* $C(x_3, y_3, z_3)$ *is*

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

PROOF Let D be the midpoint of BC. Join A and D.



Then, the coordinates of D are

$$\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right).$$

Let G be the centroid of triangle ABC. Then, G lies on AD and divides it in the ratio 2 : 1.

$\therefore$ the coordinates of G are

$$\left[\frac{2\cdot\left(\frac{x_2 + x_3}{2}\right) + 1\cdot x_1}{2+1}, \frac{2\cdot\left(\frac{y_2 + y_3}{2}\right) + 1\cdot y_1}{2+1}, \frac{2\cdot\left(\frac{z_2 + z_3}{2}\right) + 1\cdot z_1}{2+1}\right],$$

i.e., $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the coordinates of the point which divides the join of the points* $P(5, 4, 2)$ *and* $Q(-1, -2, 4)$ *in the ratio* 2 : 3.

SOLUTION Let $R(x, y, z)$ be the required point. Then,

$$x = \frac{2(-1) + 3\times 5}{2+3}, \; y = \frac{2(-2) + 3\times 4}{2+3}, \; z = \frac{2\times 4 + 3\times 2}{2+3}$$

or $x = \frac{13}{5}$, $y = \frac{8}{5}$ and $z = \frac{14}{5}$.

So, the required point is $R\left(\frac{13}{5}, \frac{8}{5}, \frac{14}{5}\right)$.

EXAMPLE 2 *Find the coordinates of the point which divides the join of the points* $A(2, -1, 3)$ *and* $B(4, 3, 1)$ *externally in the ratio* 3 : 4.

SOLUTION Let $C(x, y, z)$ be the required point. Then,

$$x = \frac{3 \times 4 - 4 \times 2}{3 - 4}, \; y = \frac{3 \times 3 - 4(-1)}{3 - 4}, \; \frac{3 \times 1 - 4 \times 3}{3 - 4}$$

or $x = -4, \; y = -13$ and $z = 9$.

$\therefore$ the required point is $C(-4, -13, 9)$.

EXAMPLE 3 *Find the ratio in which the join of the points* $P(2, -1, 3)$ *and* $Q(4, 3, 1)$ *is divided by the point* $\left(\frac{20}{7}, \frac{5}{7}, \frac{15}{7}\right)$.

SOLUTION Let the required ratio be $\lambda : 1$. Then, the coordinates of R are

$$\left(\frac{4\lambda + 2}{\lambda + 1}, \frac{3\lambda - 1}{\lambda + 1}, \frac{\lambda + 3}{\lambda + 1}\right).$$

But, the coordinates of R are $\left(\frac{20}{7}, \frac{5}{7}, \frac{15}{7}\right)$

$\therefore \quad \frac{4\lambda + 2}{\lambda + 1} = \frac{20}{7}$ or $\lambda = \frac{3}{4}$.

So, the required ratio is $\frac{3}{4} : 1$, i.e., $3 : 4$.

EXAMPLE 4 *Find the ratio in which the line segment, joining the points* $P(2, 3, 4)$ *and* $Q(-3, 5, -4)$ *is divided by the yz-plane. Also, find the point of intersection.*

SOLUTION Let PQ be the divided by the yz-plane at a point R in the ratio $\lambda : 1$. Then, the coordinates of R are

$$\left(\frac{-3\lambda + 2}{\lambda + 1}, \frac{5\lambda + 3}{\lambda + 1}, \frac{-4\lambda + 4}{\lambda + 1}\right). \qquad \text{... (i)}$$

Since R lies on the yz-plane, the x-coordinate or R is therefore 0.

$\therefore \quad \frac{-3\lambda + 2}{\lambda + 1} = 0, \quad$ or $\quad \lambda = \frac{2}{3}$.

So, the required ratio is $\frac{2}{3} : 1$, i.e., $2 : 3$.

Putting $\lambda = \frac{2}{3}$ in (i), the point of intersection of the line segment PQ and the yz-plane is $\left(0, \frac{19}{5}, \frac{4}{5}\right)$.

EXAMPLE 5 *Find the ratio in which the join of* $A(2, 1, 5)$ *and* $B(3, 4, 3)$ *is divided by the plane* $2x + 2y - 2z = 1$. *Also, find the coordinates of the point of division.*

SOLUTION Suppose the given plane intersects AB at a point C and let the required ratio be $\lambda : 1$.

Then, the coordinates of C are

$$\left(\frac{3\lambda+2}{\lambda+1}, \frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+5}{\lambda+1}\right). \qquad \ldots (1)$$

Since C lies on the plane $2x + 2y - 2z = 1$, this point must satisfy the equation of the plane.

$$\therefore \quad 2\left(\frac{3\lambda+2}{\lambda+1}\right) + 2\left(\frac{4\lambda+1}{\lambda+1}\right) - 2\left(\frac{3\lambda+5}{\lambda+1}\right) = 1 \quad \text{or} \quad \lambda = \frac{5}{7}.$$

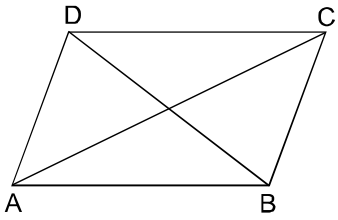
So, the required ratio is $\frac{5}{7} : 1$, i.e., $5 : 7$.

Putting $\lambda = \frac{5}{7}$ in (i), the required point of division is $C\left(\frac{29}{12}, \frac{9}{4}, \frac{25}{6}\right)$.

EXAMPLE 6 *Three vertices of a parallelogram ABCD are* $A(3, -1, 2)$, $B(1, 2, -4)$ *and* $C(-1, 1, 2)$. *Find the coordinates of the fourth vertex D.*

SOLUTION Let $D(x, y, z)$ be the required point. Then, the midpoint of diagonal BD is

$$\left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-4}{2}\right).$$



And, the midpoint of diagonal AC is

$$\left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right), \text{ i.e., } (1, 0, 2).$$

But, the midpoints of the diagonals of a parallelogram always coincide.

$$\therefore \quad \frac{x+1}{2} = 1, \frac{y+2}{2} = 0 \text{ and } \frac{z-4}{2} = 2.$$

So, $x = 1$, $y = -2$ and $z = 8$.

Hence, the required point is $D(1, -2, 8)$.

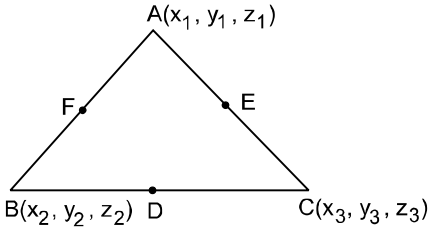
EXAMPLE 7 *The midpoints of the sides of a triangle are* $(1, 5, -1)$, $(0, 4, -2)$ *and* $(2, 3, 4)$. *Find its vertices.*

SOLUTION Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of the given triangle, and let $D(1, 5, -1)$, $E(0, 4, -2)$ and $F(2, 3, 4)$ be the midpoints of the sides BC, CA and AB respectively. Then,

$$\frac{x_2+x_3}{2} = 1; \frac{y_2+y_3}{2} = 5; \frac{z_2+z_3}{2} = 1;$$

$$\frac{x_3+x_1}{2} = 0; \frac{y_3+y_1}{2} = 4; \frac{z_3+z_1}{2} = -2;$$

$$\frac{x_1+x_2}{2} = 2; \frac{y_1+y_2}{2} = 3 \text{ and } \frac{z_1+z_2}{2} = 4.$$



Thus, $x_2 + x_3 = 2$; $x_3 + x_1 = 0$; $x_1 + x_2 = 4$;

$y_2 + y_3 = 10$; $y_3 + y_1 = 8$; $y_1 + y_2 = 6$;

$z_2 + z_3 = -2$; $z_3 + z_1 = -4$; $z_1 + z_2 = 8$.

Adding first three equations, we get

$$2(x_1 + x_2 + x_3) = 6 \quad \text{or} \quad x_1 + x_2 + x_3 = 3.$$

Thus, $x_1 = 1,\ x_2 = 3$ and $x_3 = -1$.

Adding next three equations, we get

$$2(y_1 + y_2 + y_3) = 24 \quad \text{or} \quad y_1 + y_2 + y_3 = 12.$$

$\therefore$ $y_1 = 2;\ y_2 = 4$ and $y_3 = 6$.

Adding last three equations, we get

$$2(z_1 + z_2 + z_3) = 2 \quad \text{or} \quad z_1 + z_2 + z_3 = 1.$$

$\therefore$ $z_1 = 3,\ z_2 = 5$ and $z_3 = -7$.

Hence, the vertices of the given triangle are

$A(1, 2, 3);\ B(3, 4, 5)$ and $C(-1, 6, -7)$.

EXAMPLE 8 *Using the section formula, prove that the three points* $A(-2, 3, 5),\ B(1, 2, 3)$ *and* $C(7, 0, -1)$ *are collinear.*

SOLUTION Suppose C divides AB in the ratio $\lambda : 1$.

Then, $\dfrac{\lambda - 2}{\lambda + 1} = 7,\ \dfrac{2\lambda + 3}{\lambda + 1} = 0$ and $\dfrac{3\lambda + 5}{\lambda + 1} = -1.$

From each of these equations, we get $\lambda = \dfrac{-3}{2}$.

This shows that C divides AB externally in the ratio $3 : 2$.

So, C lies on the line joining A and B.

Hence, the given points A, B, C are collinear.

EXERCISE 26C

1. Find the coordinates of the point which divides the join of $A(3, 2, 5)$ and $B(-4, 2, -2)$ in the ratio $4 : 3$.

2. Let $A(2, 1, -3)$ and $B(5, -8, 3)$ be two given points. Find the coordinates of the points of trisection of the line segment AB.

3. Find the coordinates of the point that divides the join of $A(-2, 4, 7)$ and $B(3, -5, 8)$ extremally in the ratio $2 : 1$.

4. Find the ratio in which the point $R(5, 4, -6)$ divides the join of $P(3, 2, -4)$ and $Q(9, 8, -10)$.

5. Find the ratio in which the point $C(5, 9, -14)$ divides the join of $A(2, -3, 4)$ and $B(3, 1, -2)$.

6. Find the ratio in which the line segment having the end points $A(-1, -3, 4)$ and $B(4, 2, -1)$ is divided by the xz-plane. Also, find the coordinates of the point of division.

7. Find the coordinates of the point where the line joining $A(3, 4, 1)$ and $B(5, 1, 6)$ crosses the xy-plane.

8. Find the ratio in which the plane $x - 2y + 3z = 5$ divides the join of $A(3, -5, 4)$ and $B(2, 3, -7)$. Find the coordinates of the point of intersection of the line and the plane.

9. The vertices of a triangle ABC are $A(3, 2, 0)$, $B(5, 3, 2)$ and $C(-9, 6, -3)$. The bisector AD of $\angle A$ meets BC at D. Find the coordinates of D.

10. If the three consecutive vertices of a parallelogram be $A(3, 4, -1)$, $B(7, 10, -3)$ and $C(5, -2, 7)$, find the fourth vertex D.

11. Two vertices of a triangle ABC are $A(2, -4, 3)$ and $B(3, -1, -2)$, and its centroid is $(1, 0, 3)$. Find its third vertex C.

12. If origin is the centroid of traingle ABC with vertices $A(a, 1, 3)$, $B(-2, b, -5)$ and $C(4, 7, c)$, find the values of a, b, c.

13. The midpoints of the sides of a triangle are $(1, 5, -1)$, $(0, 4, -2)$ and $(2, 3, 4)$. Find its vertices.

ANSWERS (EXERCISE 26C)

1. $(-1, 2, 1)$ **2.** $(3, -2, -1), (4, -5, 1)$ **3.** $(8, -14, 9)$

4. $1 : 2$ **5.** $3 : 2$ (externally) **6.** $3 : 2, (2, 0, 1)$

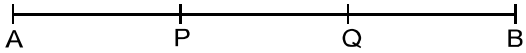
7. $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$ **8.** $2 : 3, \left(\frac{13}{5}, \frac{-9}{5}, \frac{-2}{5}\right)$ **9.** $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$

10. $(1, -8, 9)$ **11.** $(-2, 5, 8)$ **12.** $a = -2, b = -8, c = 2$

13. $(1, 2, 3); (3, 4, 5); (-1, 6, -7)$

HINTS TO SOME SELECTED QUESTIONS

2. Let P and Q be the points of trisection of AB. Then, P divides AB in the ratio $1 : 2$ and Q divides AB in the ratio $2 : 1$.



7. Suppose the xy-plane divides AB in the ratio $\lambda : 1$.

Then, $\frac{6\lambda + 1}{\lambda + 1} = 0 \Rightarrow \lambda = -\frac{1}{6}$.

Divide AB in the ratio $1 : (-6)$.

8. D divides BC in the ratio $AB : AC$.

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Limits

CONCEPT OF LIMIT

Consider the function $f(x) = \dfrac{(x^2 - 1)}{(x - 1)}$.

Clearly, $f(1) = \dfrac{0}{0}$, which is meaningless.

Thus, $f(x)$ is not defined at $x = 1$.

Now, $f(x) = \dfrac{(x^2 - 1)}{(x - 1)} = \dfrac{(x - 1)\,(x + 1)}{(x - 1)} = (x + 1)$, only when $x \neq 1$.

If we give to x, a value not exactly 1 but slightly more than 1 then clearly, the value of $f(x)$ is slightly more than 2. Now, if we go on decreasing this value and take it nearer to 1 then clearly the value of $f(x)$ will come nearer to 2, as shown below.

If $x = 1.1$	then $f(x) = 2.1$.
If $x = 1.01$	then $f(x) = 2.01$.
If $x = 1.001$	then $f(x) = 2.001$.
… … …	… … …
If $x = 1.0000001$	then $f(x) = 2.0000001$.
… … …	… … …

Thus, as the value of x approaches 1, the value of $f(x)$ approaches 2, and we write:

as $x \to 1, f(x) \to 2$.

(The symbol '$\to$' stands for 'approaches to'.)

Similarly, if we give to x, a value slightly less than 1 then the value of $f(x)$ is slightly less than 2. Now, if we go on increasing this value and take it nearer to 1, the value of $f(x)$ will come nearer to 2, as shown below.

If $x = 0.9$	then $f(x) = 1.9$.
If $x = 0.99$	then $f(x) = 1.99$.

If $x = 0.999$ then $f(x) = 1.999$.

… … … … … …

If $x = 0.999999$ then $f(x) = 1.999999$.

… … … … … …

Thus, in this case also

as $x \to 1, f(x) \to 2$.

We express this fact as $\lim_{x \to 1} \left(\frac{x^2 - 1}{x - 1} \right) = 2$.

LIMIT *We say that* $\lim_{x \to a} f(x) = l$ *if whenever* $x \to a, f(x) \to l$.

Working rules for finding $\lim_{x \to a} f(x)$

RULE I Put $x = a$ in the given function. If $f(a)$ is a definite value then

$$\lim_{x \to a} f(x) = f(a).$$

EXAMPLES 1. $\lim_{x \to 0} \sin x = \sin 0 = 0$.

2. $\lim_{x \to 1} (x^2 + 5x - 2) = (1^2 + 5 \times 1 - 2) = 4$.

If $f(a)$ is indeterminate, we adopt the rules given below.

RULE II If $f(x)$ is a rational function then factorize the numerator and the denominator. Cancel out the common factors and then put $x = a$.

EXAMPLE *Evaluate* (i) $\lim_{x \to 3} \left(\frac{x^2 - 9}{x - 3} \right)$ (ii) $\lim_{x \to 1} \frac{(x^2 - 4x + 3)}{(x - 1)}$.

SOLUTION (i) $\lim_{x \to 3} \left(\frac{x^2 - 9}{x - 3} \right) = \lim_{x \to 3} \frac{(x + 3)(x - 3)}{(x - 3)} = \lim_{x \to 3} (x + 3) = 6$.

(ii) $\lim_{x \to 1} \frac{(x^2 - 4x + 3)}{(x - 1)} = \lim_{x \to 1} \frac{(x - 1)(x - 3)}{(x - 1)} = \lim_{x \to 1} (x - 3) = -2$.

RULE III If the given function contains a surd then simplify it by using conjugate surds. After simplification, put $x = a$.

EXAMPLE *Evaluate* $\lim_{x \to 0} \left\{ \frac{\sqrt{1 + x} - \sqrt{1 - x}}{x} \right\}$.

SOLUTION We have

$$\lim_{x \to 0} \left\{ \frac{\sqrt{1 + x} - \sqrt{1 - x}}{x} \right\} = \lim_{x \to 0} \left\{ \frac{(\sqrt{1 + x} - \sqrt{1 - x}}{x} \cdot \frac{(\sqrt{1 + x} + \sqrt{1 - x})}{(\sqrt{1 + x} + \sqrt{1 - x})} \right\}$$

$$= \lim_{x \to 0} \frac{\{(1+x)-(1-x)\}}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x \to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x \to 0} \frac{2}{(\sqrt{1+x}+\sqrt{1-x})} = 1 \qquad \text{[putting } x = 0\text{]}.$$

RULE IV If the given function contains a series which is capable of being expanded then after making proper expansion and simplifying, cancel the common factors in the numerator and denominator, if any. Then, put $x = a$.

Some important expansions are given below:

(i) For $|x| < 1$, we have the binomial expansion

$$(1+x)^n = \left\{1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\right\}$$

(ii) $$\left(\frac{x^n - a^n}{x-a}\right) = (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1})$$

(iii) $$e^x = \left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots\right\}$$

(iv) $$a^x = \left\{1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \ldots\right\}$$

(v) $$\log(1+x) = \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right\}$$

(vi) $$\sin x = \left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right\}$$

(vii) $$\cos x = \left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right\}$$ (viii) $$\tan x = \left\{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right\}$$

We shall also make use of the following theorems on limits. The proof of these theorems is beyond the scope of this book.

Fundamental Theorems on Limits (without proof)

(i) $$\lim_{x \to a}\{f(x)+g(x)\} = \left\{\lim_{x \to a} f(x)\right\} + \left\{\lim_{x \to a} g(x)\right\}$$

(ii) $$\lim_{x \to a}\{f(x)-g(x)\} = \left\{\lim_{x \to a} f(x)\right\} - \left\{\lim_{x \to a} g(x)\right\}$$

(iii) $$\lim_{x \to a}\{c \cdot f(x)\} = c \cdot \left\{\lim_{x \to a} f(x)\right\},$$ where c is a constant

(iv) $$\lim_{x \to a}\{f(x) \cdot g(x)\} = \left\{\lim_{x \to a} f(x)\right\} \cdot \left\{\lim_{x \to a} g(x)\right\}$$

(v) $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\}=\frac{\left\{\lim_{x\to a} f(x)\right\}}{\left\{\lim_{x\to a} g(x)\right\}}$, provided $\lim_{x\to a} g(x)\neq 0$

(vi) If $f(x)\leq g(x)$ for all x then $\lim_{x\to a} f(x)\leq \lim_{x\to a} g(x)$.

Things to Remember

(i) $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(iii) $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Some more Important Theorems on Limits

THEOREM 1 $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)=na^{n-1}$, *where* $a>0$.

PROOF We know that $\left(\frac{x^n-a^n}{x-a}\right)=(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-1})$.

$$\therefore \lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)=\lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-1})$$

$$=na^{n-1} \qquad [\text{putting } x=a].$$

THEOREM 2 $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=1.$

PROOF We have

$$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=\lim_{x\to 0}\left\{\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)-1}{x}\right\}\left\{\because\ e^x=\left(1+x+\frac{x^2}{2!}+\ldots\right)\right\}$$

$$=\lim_{x\to 0}\left\{\frac{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)}{x}\right\}=\lim_{x\to 0}\left\{\frac{x\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots\right)}{x}\right\}$$

$$=\lim_{x\to 0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots\right)$$

$$=1 \qquad [\text{putting } x=0].$$

THEOREM 3 $\lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log a.$

PROOF $\lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\lim_{x\to 0}\left\{\frac{\left\{1+x(\log a)+\frac{x^2}{2!}(\log a)^2+\ldots\right\}-1}{x}\right\}$ [on expanding a^x]

$$=\lim_{x\to 0}\frac{x\left\{(\log a)+\frac{x}{2!}(\log a)^2+\ldots\right\}}{x}$$

$$=\lim_{x\to 0}\left\{(\log a)+\frac{x}{2!}(\log a)^2+\frac{x^2}{3!}(\log a)^3+\ldots\right\}$$

$$=\log a \qquad [\text{putting } x=0].$$

THEOREM 4 $\lim_{x\to 0}(1+x)^{1/x}=e.$

PROOF $\lim_{x\to 0}(1+x)^{1/x}=\lim_{x\to 0}\left\{1+\frac{1}{x}\cdot x+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}\cdot x^2+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}\cdot x^3+\ldots\infty\right\}$

[using the binomial expansion of $(1+x)^{1/x}$]

$$=\lim_{x\to 0}\left\{1+1+\frac{(1-x)}{2!}+\frac{(1-x)(1-2x)}{3!}+\ldots\right\}$$

$$=\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots\right) \qquad [\text{putting } x=0]$$

$$=e.$$

THEOREM 5 $\lim_{x\to 0}\frac{\log(1+x)}{x}=1.$

PROOF $\lim_{x\to 0}\frac{\log(1+x)}{x}=\lim_{x\to 0}\frac{\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)}{x}$ [expanding $\log(1+x)$]

$$=\lim_{x\to 0}\left(1-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)=1 \qquad [\text{putting } x=0].$$

SUMMARY

1. $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)=na^{n-1}$, where $a>0$.
2. $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=1.$
3. $\lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log_e a.$
4. $\lim_{x\to 0}(1+x)^{1/x}=e.$
5. $\lim_{x\to 0}\frac{\log(1+x)}{x}=1.$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate* $\lim_{x \to a} \left\{ \frac{x^{12} - a^{12}}{x - a} \right\}$.

SOLUTION $\lim_{x \to a} \left\{ \frac{x^{12} - a^{12}}{x - a} \right\} = 12a^{(12-1)} = 12a^{11}$ $\left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1} \right]$.

EXAMPLE 2 *Evaluate* $\lim_{x \to 2} \left(\frac{x^5 - 32}{x^3 - 8} \right)$.

SOLUTION $\lim_{x \to 2} \left(\frac{x^5 - 32}{x^3 - 8} \right) = \lim_{x \to 2} \left(\frac{x^5 - 2^5}{x^3 - 2^3} \right)$

$= \lim_{x \to 2} \left\{ \left(\frac{x^5 - 2^5}{x - 2} \right) \div \left(\frac{x^3 - 2^3}{x - 2} \right) \right\}$

$= \lim_{x \to 2} \left(\frac{x^5 - 2^5}{x - 2} \right) \div \lim_{x \to 2} \left(\frac{x^3 - 2^3}{x - 2} \right)$

$= [(5 \times 2^4) \div (3 \times 2^2)]$ $\left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1} \right]$

$= \left(\frac{80}{12} \right) = \frac{20}{3}$.

EXAMPLE 3 *Evaluate* $\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1}$.

SOLUTION $\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1} = \lim_{x \to 1} \left\{ \left(\frac{x^{15} - 1}{x - 1} \right) \div \left(\frac{x^{10} - 1}{x - 1} \right) \right\}$

$= \lim_{x \to 1} \left(\frac{x^{15} - 1}{x - 1} \right) \div \lim_{x \to 1} \left(\frac{x^{10} - 1}{x - 1} \right)$

$= (15 \times 1^{14}) \div (10 \times 1^9) = \frac{15}{10} = \frac{3}{2}$.

EXAMPLE 4 *Evaluate* $\lim_{x \to a} \left(\frac{x^m - a^m}{x^n - a^n} \right)$.

SOLUTION $\lim_{x \to a} \left(\frac{x^m - a^m}{x^n - a^n} \right) = \lim_{x \to a} \left\{ \left(\frac{x^m - a^m}{x - a} \right) \div \left(\frac{x^n - a^n}{x - a} \right) \right\}$

$= \lim_{x \to a} \left(\frac{x^m - a^m}{x - a} \right) \div \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right)$

$= (ma^{m-1}) \ (na^{n-1})$

$= \frac{ma^{m-1}}{na^{n-1}} = \frac{m}{n} a^{m-n}$.

EXAMPLE 5 *Evaluate* $\lim_{x\to a}\dfrac{(x+2)^{3/2}-(a+2)^{3/2}}{x-a}$.

SOLUTION $\lim_{x\to a}\dfrac{(x+2)^{3/2}-(a+2)^{3/2}}{x-a}$

$$=\lim_{(x+2)\to(a+2)}\frac{(x+2)^{3/2}-(a+2)^{3/2}}{(x+2)-(a+2)}$$

$$=\frac{3}{2}\cdot(a+2)^{\left(\frac{3}{2}-1\right)}=\frac{3}{2}(a+2)^{1/2}\qquad\left[\because\ \lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)=na^{n-1}\right].$$

EXAMPLE 6 *Evaluate* $\lim_{x\to 0}\dfrac{(1+x)^n-1}{x}$.

SOLUTION Put $(1+x)=y$, so that when $x\to 0$ then $y\to 1$.

$$\therefore\quad\lim_{x\to 0}\frac{(1+x)^n-1}{x}=\lim_{y\to 1}\left(\frac{y^n-1}{y-1}\right)$$

$$=\lim_{y\to 1}\left(\frac{y^n-1^n}{y-1}\right)=n\times 1^{(n-1)}=(n\times 1)=n.$$

EXAMPLE 7 *Evaluate* $\lim_{x\to 1}\left(\dfrac{1-x^{-1/3}}{1-x^{-2/3}}\right)$.

SOLUTION $\lim_{x\to 1}\left(\dfrac{1-x^{-1/3}}{1-x^{-2/3}}\right)=\lim_{x\to 1}\left[\dfrac{(1-x^{-1/3})}{(1)^2-(x^{-1/3})^2}\right]$

$$=\lim_{x\to 1}\frac{(1-x^{-1/3})}{(1-x^{-1/3})(1+x^{-1/3})}$$

$$=\lim_{x\to 1}\frac{1}{(1+x^{-1/3})}=\frac{1}{\{1+(1)^{-1/3}\}}=\frac{1}{2}.$$

EXAMPLE 8 *Evaluate* $\lim_{x\to 0}\dfrac{(\sqrt{1+3x}-\sqrt{1-3x})}{x}$. **[CBSE 2000C]**

SOLUTION $\lim_{x\to 0}\dfrac{(\sqrt{1+3x}-\sqrt{1-3x})}{x}$

$$=\lim_{x\to 0}\left\{\frac{(\sqrt{1+3x}-\sqrt{1-3x})}{x}\times\frac{(\sqrt{1+3x}+\sqrt{1-3x})}{(\sqrt{1+3x}+\sqrt{1-3x})}\right\}$$

$$=\lim_{x\to 0}\frac{\{(1+3x)-(1-3x)}{x(\sqrt{1+3x}+\sqrt{1-3x})}=\lim_{x\to 0}\frac{6x}{x(\sqrt{1+3x}+\sqrt{1-3x})}$$

$$=\lim_{x\to 0}\frac{6}{(\sqrt{1+3x}+\sqrt{1-3x})}=\frac{6}{(\sqrt{1}+\sqrt{1})}=\frac{6}{2}=3.$$

EXAMPLE 9 *Evaluate* $\lim_{x\to 2}\left\{\dfrac{(x^2-4)}{\sqrt{3x-2}-\sqrt{x+2}}\right\}$.

SOLUTION $\lim_{x\to 2}\left\{\dfrac{(x^2-4)}{\sqrt{3x-2}-\sqrt{x+2}}\right\}$

$$= \lim_{x\to 2}\left\{\frac{(x^2-4)}{(\sqrt{3x-2}-\sqrt{x+2})}\times\frac{(\sqrt{3x-2}+\sqrt{x-2})}{(\sqrt{3x-2}+\sqrt{x-2})}\right\}$$

$$= \lim_{x\to 2}\left\{\frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x-2})}{(3x-2)-(x+2)}\right\}$$

$$= \lim_{x\to 2}\left\{\frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x-2})}{2(x-2)}\right\}$$

$$= \lim_{x\to 2}\frac{(x+2)(\sqrt{3x-2}+\sqrt{x-2})}{2} \qquad [\text{cancelling } (x-2)]$$

$$= 8 \qquad [\text{putting } x=2].$$

EXAMPLE 10 *Evaluate* $\lim_{x\to 0}\left(\dfrac{e^{3x}-1}{x}\right)$.

SOLUTION $\lim_{x\to 0}\left(\dfrac{e^{3x}-1}{x}\right) = \lim_{3x\to 0}\left\{\left(\dfrac{e^{3x}-1}{3x}\right)\times 3\right\}$ $\qquad [\because\ (x\to 0)\ \Rightarrow\ (3x\to 0)]$

$$= 3\times\lim_{y\to 0}\left(\frac{e^y-1}{y}\right), \quad \text{where } y=3x$$

$$= (3\times 1) = 3 \qquad \left[\because\ \lim_{y\to 0}\left(\frac{e^y-1}{y}\right)=1\right].$$

EXAMPLE 11 *Evaluate* $\lim_{x\to 2}\left(\dfrac{e^x-e^2}{x-2}\right)$. **[CBSE 2003]**

SOLUTION Put $(x-2)=y$, so that when $x\to 2$ then $y\to 0$.

$$\therefore \lim_{x\to 2}\left(\frac{e^x-e^2}{x-2}\right) = \lim_{y\to 0}\left(\frac{e^{y+2}-e^2}{y}\right)$$

$$= \lim_{y\to 0}\left\{e^2\cdot\left(\frac{e^y-1}{y}\right)\right\}$$

$$= e^2\cdot\lim_{y\to 0}\left(\frac{e^y-1}{y}\right) = (e^2\times 1) = e^2\left[\because\ \lim_{y\to 0}\left(\frac{e^y-1}{y}\right)=1\right].$$

EXAMPLE 12 *Evaluate:*

(i) $\lim_{x\to 0}\left(\dfrac{e^{-x}-1}{x}\right)$ *(ii)* $\lim_{x\to 0}\left(\dfrac{e^x-e^{-x}}{x}\right)$ *(iii)* $\lim_{x\to 0}\left(\dfrac{e^x+e^{-x}-2}{x^2}\right)$ **[CBSE 2004]**

SOLUTION (i) $\lim_{x\to 0}\left(\frac{e^{-x}-1}{x}\right)=\lim_{y\to 0}\left(\frac{e^{y}-1}{-y}\right)$, where $-x=y$

$$=-\lim_{y\to 0}\left(\frac{e^{y}-1}{y}\right)=-1 \qquad \left[\because \lim_{y\to 0}\left(\frac{e^{y}-1}{y}\right)=1\right].$$

(ii) $\lim_{x\to 0}\left(\frac{e^{x}-e^{-x}}{x}\right)=\lim_{x\to 0}\left\{\frac{(e^{x}-1)-(e^{-x}-1)}{x}\right\}$

$$=\lim_{x\to 0}\left\{\left(\frac{e^{x}-1}{x}\right)-\left(\frac{e^{-x}-1}{x}\right)\right\}$$

$$=\lim_{x\to 0}\left(\frac{e^{x}-1}{x}\right)-\lim_{x\to 0}\left(\frac{e^{-x}-1}{x}\right)$$

$$=1-\lim_{y\to 0}\left(\frac{e^{y}-1}{-y}\right),$$

where $-x=y\left[\because \lim_{x\to 0}\left(\frac{e^{x}-1}{x}\right)=1\right]$

$$=1+\lim_{y\to 0}\left(\frac{e^{y}-1}{y}\right)=(1+1)=2\left[\because \lim_{y\to 0}\left(\frac{e^{y}-1}{y}\right)=1\right].$$

(iii) $\lim_{x\to 0}\left(\frac{e^{x}+e^{-x}-2}{x^{2}}\right)=\lim_{x\to 0}\left(\frac{e^{2x}+1-2e^{x}}{x^{2}e^{x}}\right)$

$$=\lim_{x\to 0}\left\{\left(\frac{e^{x}-1}{x}\right)^{2}\cdot\frac{1}{e^{x}}\right\}$$

$$=\lim_{x\to 0}\left(\frac{e^{x}-1}{x}\right)^{2}\cdot\lim_{x\to 0}\frac{1}{e^{x}}$$

$$=\left(1^{2}\times\frac{1}{e^{0}}\right)=(1\times 1)=1 \qquad \left[\because \lim_{x\to 0}\left(\frac{e^{x}-1}{x}\right)=1\right].$$

EXAMPLE 13 *Evaluate* $\lim_{x\to 0}\left(\frac{3^{x}-2^{x}}{x}\right)$.

SOLUTION $\lim_{x\to 0}\left(\frac{3^{x}-2^{x}}{x}\right)=\lim_{x\to 0}\left\{\frac{(3^{x}-1)-(2^{x}-1)}{x}\right\}$

$$=\lim_{x\to 0}\left(\frac{3^{x}-1}{x}\right)-\lim_{x\to 0}\left(\frac{2^{x}-1}{x}\right)$$

$$=(\log\ 3-\log\ 2)=\log\frac{3}{2} \qquad \left[\because \lim_{x\to 0}\left(\frac{a^{x}-1}{x}\right)=\log a\right].$$

EXAMPLE 14 *Evaluate* $\lim_{x\to 0}\left(\frac{3^{2x}-2^{3x}}{x}\right)$.

SOLUTION $\lim_{x\to 0}\left(\frac{3^{2x}-2^{3x}}{x}\right)=\lim_{x\to 0}\left\{\frac{(3^{2x}-1)-(2^{3x}-1)}{x}\right\}$

$$=\lim_{x\to 0}\left(\frac{3^{2x}-1}{x}\right)-\lim_{x\to 0}\left(\frac{2^{3x}-1}{x}\right)$$

$$=\lim_{x\to 0}\left\{\left(\frac{3^{2x}-1}{2x}\right)\cdot 2\right\}-\lim_{x\to 0}\left\{\left(\frac{2^{3x}-1}{3x}\right)\cdot 3\right\}$$

$$=2\cdot\lim_{2x\to 0}\left(\frac{3^{2x}-1}{2x}\right)-3\cdot\lim_{3x\to 0}\left(\frac{2^{3x}-1}{3x}\right)$$

$$=(2\log\ 3-3\log\ 2)\qquad\left[\because\ \lim_{y\to 0}\left(\frac{a^y-1}{y}\right)=\log\ a\right]$$

$$=[\log\ (3)^2-\log\ (2)^3]=(\log\ 9-\log\ 8)=\log\left(\frac{9}{8}\right).$$

EXAMPLE 15 *Evaluate* $\lim_{x\to 0}\left(\frac{3^{2x}-1}{2^{3x}-1}\right)$.

SOLUTION $\lim_{x\to 0}\left(\frac{3^{2x}-1}{2^{3x}-1}\right)=\lim_{x\to 0}\frac{\left\{\left(\frac{3^{2x}-1}{2x}\right)\cdot 2x\right\}}{\left\{\left(\frac{2^{3x}-1}{3x}\right)\cdot 3x\right\}}$

$$=\frac{2}{3}\cdot\frac{\lim_{2x\to 0}\left(\frac{3^{2x}-1}{2x}\right)}{\lim_{3x\to 0}\left(\frac{2^{3x}-1}{3x}\right)}$$

$$=\frac{2}{3}\cdot\frac{\log\ 3}{\log\ 2}\qquad\left[\because\ \lim_{y\to 0}\left(\frac{a^y-1}{y}\right)=\log\ a\right]$$

$$=\frac{2\log\ 3}{3\log\ 2}=\frac{\log\ (3^2)}{\log\ (2^3)}=\frac{\log\ 9}{\log\ 8}.$$

EXAMPLE 16 *Evaluate* $\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$.

SOLUTION $\lim_{x\to 1}\frac{(x^2-\sqrt{x})}{(\sqrt{x}-1)}=\lim_{x\to 1}\left\{\frac{(x^2-\sqrt{x}}{(\sqrt{x}-1)}\times\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}\times\frac{(x^2+\sqrt{x})}{(x^2+\sqrt{x})}\right\}$

$$=\lim_{x\to 1}\frac{(x^4-x)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})}=\lim_{x\to 1}\frac{x(x^3-1)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})}$$

$$= \lim_{x\to 1} \frac{x(x-1)(x^2+x+1)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})}$$

$$= \lim_{x\to 1} \frac{x(x^2+x+1)(\sqrt{x}+1)}{(x^2+\sqrt{x})}$$

$$= \frac{1\times(1^2+1+1)(\sqrt{1}+1)}{(1^2+\sqrt{1})} = \left(\frac{1\times 3\times 2}{2}\right) = 3 \quad [\text{putting } x=1].$$

EXAMPLE 17 *Evaluate* $\lim_{x\to a} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$.

SOLUTION $\lim_{x\to a} \dfrac{(\sqrt{a+2x}-\sqrt{3x})}{(\sqrt{3a+x}-2\sqrt{x})}$

$$= \lim_{x\to a} \frac{(\sqrt{a+2x}-\sqrt{3x})}{(\sqrt{3a+x}-2\sqrt{x})} \times \frac{(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{3a+x}+2\sqrt{x})} \times \frac{(\sqrt{a+2x}+\sqrt{3x})}{(\sqrt{a+2x}+\sqrt{3x})}$$

$$\lim_{x\to a} \frac{[(a+2x)-3x]\times(\sqrt{3a+x}+2\sqrt{x})}{[(3a+x)-4x]\times(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x\to a} \frac{(a-x)\times(\sqrt{3a+x}+2\sqrt{x})}{3(a-x)\times(\sqrt{a+2x}+\sqrt{3x})} = \lim_{x\to a} \frac{(\sqrt{3a+x}+2\sqrt{x})}{3(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \frac{(\sqrt{4a}+2\sqrt{a})}{3(\sqrt{3a}+\sqrt{3a})} = \frac{4\sqrt{a}}{6\sqrt{3}\sqrt{a}} = \frac{2}{3\sqrt{3}}.$$

EXAMPLE 18 *Evaluate* $\lim_{x\to 1} \dfrac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}$. **[CBSE 2005C]**

SOLUTION $\lim_{x\to 1} \dfrac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)} = \lim_{x\to 1} \dfrac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$

$$= \lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)(\sqrt{x}+1)}{(2x+3)(x-1)(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{(2x-3)(x-1)}{(2x+3)(x-1)(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{(2x-3)}{(2x+3)(\sqrt{x}+1)} = \frac{(2\times 1-3)}{(2\times 1+3)(\sqrt{1}+1)} = \frac{-1}{10}.$$

EXAMPLE 19 *Evaluate* $\lim_{x\to 0} \dfrac{(1+x)^4-1}{x}$.

SOLUTION Put $(1+x)=y$, so that when $x\to 0$ then $y\to 1$.

$$\therefore \quad \lim_{x\to 0}\frac{(1+x)^4-1}{x}=\lim_{y\to 1}\left(\frac{y^4-1}{y-1}\right)=\lim_{y\to 1}\left(\frac{y^4-1^4}{y-1}\right)$$

$$=4\times 1^{(4-1)} \qquad \left[\because \lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)=na^{n-1}\right]$$

$$=(4\times 1^3)=4.$$

EXAMPLE 20 *Evaluate* $\lim_{x\to 0}\dfrac{(1+x)^6-1}{(1+x)^5-1}$.

SOLUTION Put $(1+x)=y$, so that when $x\to 0$ then $y\to 1$.

$$\therefore \quad \lim_{x\to 0}\frac{(1+x)^6-1}{(1+x)^5-1}$$

$$=\lim_{y\to 1}\left(\frac{y^6-1}{y^5-1}\right)=\frac{\lim_{y\to 1}\left(\frac{y^6-1}{y-1}\right)}{\lim_{y\to 1}\left(\frac{y^5-1}{y-1}\right)}=\frac{\lim_{y\to 1}\left(\frac{y^6-1^6}{y-1}\right)}{\lim_{y\to 1}\left(\frac{y^5-1^5}{y-1}\right)}=\frac{6\times 1^{(6-1)}}{5\times 1^{(5-1)}}$$

$$=\frac{6\times 1^5}{5\times 1^4}=\frac{6}{5} \qquad \left[\because \lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)=na^{n-1}\right].$$

EXERCISE 27A

Evaluate the following limits:

1. $\lim_{x\to 2}(5-x)$

2. $\lim_{x\to 1}(6x^2-4x+3)$

3. $\lim_{x\to 3}\left(\dfrac{x^2+9}{x+3}\right)$

4. $\lim_{x\to 3}\left(\dfrac{x^2-4x}{x-2}\right)$

5. $\lim_{x\to 5}\left(\dfrac{x^2-25}{x-5}\right)$

6. $\lim_{x\to 1}\left(\dfrac{x^3-1}{x-1}\right)$

7. $\lim_{x\to -2}\left(\dfrac{x^3+8}{x+2}\right)$

8. $\lim_{x\to 3}\left(\dfrac{x^4-81}{x-3}\right)$

9. $\lim_{x\to 3}\left(\dfrac{x^2-4x+3}{x^2-2x-3}\right)$

10. $\lim_{x\to \frac{1}{2}}\left(\dfrac{4x^2-1}{2x-1}\right)$

11. $\lim_{x\to 4}\left(\dfrac{x^3-64}{x^2-16}\right)$

12. $\lim_{x\to 2}\left(\dfrac{x^5-32}{x^3-8}\right)$

13. $\lim_{x\to a}\left(\dfrac{x^{5/2}-a^{5/2}}{x-a}\right)$

14. $\lim_{x\to a}\left\{\dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}\right\}$

15. $\lim_{x\to 1}\left(\frac{x^n-1}{x-1}\right)$

16. $\lim_{x\to a}\left(\frac{\sqrt{x}-\sqrt{a}}{x-a}\right)$.

17. $\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$

18. $\lim_{h\to 0}\frac{1}{h}\left\{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right\}$

19. $\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\right)$

20. $\lim_{x\to 0}\left(\frac{\sqrt{2-x}-\sqrt{2+x}}{x}\right)$

21. $\lim_{x\to 0}\left(\frac{\sqrt{1+x+x^2}-1}{x}\right)$

22. $\lim_{x\to 0}\left(\frac{\sqrt{3-x}-1}{2-x}\right)$

23. $\lim_{x\to 0}\left(\frac{2x}{\sqrt{a+x}-\sqrt{a-x}}\right)$

24. $\lim_{x\to 1}\left(\frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}\right)$

25. $\lim_{x\to 2}\left(\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\right)$

26. $\lim_{x\to 4}\left(\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\right)$

27. $\lim_{x\to 0}\left(\frac{\sqrt{a+x}-\sqrt{a}}{x\sqrt{a(a+x)}}\right)$

28. $\lim_{x\to 0}\left(\frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}\right)$

29. $\lim_{x\to 1}\left(\frac{x^4-3x^2+2}{x^3-5x^2+3x+1}\right)$

30. $\lim_{x\to 2}\left(\frac{3^x+3^{3-x}-12}{3^{3-x}-3^{x/2}}\right)$

31. $\lim_{x\to 0}\left(\frac{e^{4x}-1}{x}\right)$

32. $\lim_{x\to 0}\left(\frac{e^{2+x}-e^2}{x}\right)$

33. $\lim_{x\to 4}\left(\frac{e^x-e^4}{x-4}\right)$

34. $\lim_{x\to 0}\left(\frac{e^{3x}-e^{2x}}{x}\right)$

35. $\lim_{x\to 0}\left(\frac{e^x-x-1}{x}\right)$

36. $\lim_{x\to 0}\left(\frac{e^{bx}-e^{ax}}{x}\right)$, $0<a<b$

37. $\lim_{x\to 0}\left(\frac{a^x-b^x}{x}\right)$

38. $\lim_{x\to 0}\left(\frac{a^x-a^{-x}}{x}\right)$

39. $\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)$

40. $\lim_{x\to 0}\left(\frac{3^{2+x}-9}{x}\right)$

ANSWERS (EXERCISE 27B)

1. 3 **2.** 5 **3.** 3 **4.** –3 **5.** 10 **6.** 3 **7.** 12 **8.** 108

9. $\frac{1}{2}$ **10.** 2 **11.** 6 **12.** $\frac{20}{3}$ **13.** $\frac{5}{2}a^{3/2}$ **14.** $\frac{5}{3}(a+2)^{2/3}$ **15.** n

16. $\frac{1}{2\sqrt{a}}$ **17.** $\frac{1}{2\sqrt{x}}$ **18.** $\frac{-1}{2x^{3/2}}$ **19.** $\frac{1}{2}$ **20.** $\frac{-1}{\sqrt{2}}$ **21.** $\frac{1}{2}$ **22.** $\frac{1}{(\sqrt{3}+1)}$ **23.** $2\sqrt{a}$

24. $\frac{1}{4}$ **25.** -8 **26.** $\frac{-1}{3}$ **27.** $\frac{1}{2a^{3/2}}$ **28.** 1 **29.** $\frac{1}{2}$ **30.** $\frac{-4}{3}$ **31.** 4 **32.** e^2 **33.** e^4

34. 1 **35.** 0 **36.** $(b-a)$ **37.** $(\log a - \log b)$ **38.** $2\log a$ **39.** $\log 2$ **40.** $9\log 3$

HINTS TO SOME SELECTED QUESTIONS

18. Given limit $= \lim_{h\to 0}\left\{\frac{(\sqrt{x}-\sqrt{x+h})}{h(\sqrt{x+h})\sqrt{x}}\times\frac{(\sqrt{x}+\sqrt{x+h})}{(\sqrt{x}+\sqrt{x+h})}\right\}$

$= \lim_{h\to 0}\frac{-h}{h(\sqrt{x+h})\sqrt{x}\,(\sqrt{x}+\sqrt{x+h})}$

$= \lim_{h\to 0}\frac{-1}{(\sqrt{x+h})\sqrt{x}\,(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{2x^{3/2}}.$

26. Given limit $= \lim_{x\to 4}\left\{\frac{(3-\sqrt{5+x})}{(1-\sqrt{5-x})}\times\frac{(1+\sqrt{5-x})}{(1+\sqrt{5-x})}\times\frac{(3+\sqrt{5+x})}{(3+\sqrt{5+x})}\right\}$

$= \lim_{x\to 4}\frac{\{9-(5+x)\}(1+\sqrt{5-x})}{\{1-(5-x)\}(3+\sqrt{5+x})} = \lim_{x\to 4}\frac{(4-x)(1+\sqrt{5-x})}{-(4-x)(3+\sqrt{5+x})}$

$= \lim_{x\to 4}\frac{(1+\sqrt{5-x})}{-(3+\sqrt{5+x})} = \frac{(1+\sqrt{1})}{-(3+\sqrt{9})} = \frac{2}{-6} = -\frac{1}{3}.$

28. Given limit $= \lim_{x\to 0}\left\{\frac{(\sqrt{1+x^2}-\sqrt{1+x})}{(\sqrt{1+x^3}-\sqrt{1+x})}\times\frac{(\sqrt{1+x^3}+\sqrt{1+x})}{(\sqrt{1+x^3}+\sqrt{1+x})}\times\frac{(\sqrt{1+x^2}+\sqrt{1+x})}{(\sqrt{1+x^2}+\sqrt{1+x})}\right\}$

$= \lim_{x\to 0}\frac{\{(1+x^2)-(1+x)\}\times(\sqrt{1+x^3}+\sqrt{1+x})}{\{(1+x^3)-(1+x)\}\times(\sqrt{1+x^2}+\sqrt{1+x})}$

$= \lim_{x\to 0}\frac{x(x-1)(\sqrt{1+x^3}+\sqrt{1+x})}{x(x-1)(x+1)(\sqrt{1+x^2}+\sqrt{1+x})}$

$= \lim_{x\to 0}\frac{(\sqrt{1+x^3}+\sqrt{1+x})}{(x+1)(\sqrt{1+x^2}+\sqrt{1+x})} = \frac{(\sqrt{1}+\sqrt{1})}{1\cdot(\sqrt{1}+\sqrt{1})} = 1.$

29. Given limit $= \lim_{x\to 1}\frac{(x-1)(x+1)(x^2-2)}{(x-1)(x^2-4x-1)} = \lim_{x\to 1}\frac{(x+1)(x^2-2)}{(x^2-4x-1)} = \frac{1}{2}.$

30. Given limit $= \lim_{x\to 2}\left\{\frac{3^x+\frac{3^3}{3^x}-12}{\frac{3^3}{3^x}-3^{x/2}}\right\} = \lim_{t\to 3}\left(\frac{t^2+\frac{27}{t^2}-12}{\frac{27}{t^2}-t}\right)$ [put $3^{x/2}=t$]

$= \lim_{t\to 3}\left(\frac{t^4-12t^2+27}{27-t^3}\right) = \lim_{t\to 3}\frac{(t^2-3)(t+3)(t-3)}{(3-t)(9+3t+t^2)}$

$= \lim_{t\to 3}\frac{(t^2-3)(t+3)}{-(9+3t+t^2)} = \frac{(9-3)(3+3)}{-(9+9+9)} = \frac{-36}{27} = \frac{-4}{3}.$

33. Put $(x-4)=y$, so that when $x\to 4$ then $y\to 0$.

$$\therefore \quad \text{given limit} = \lim_{y\to 0}\left\{\frac{e^{y+4}-e^4}{y}\right\} = e^4\cdot\lim_{y\to 0}\left\{\frac{e^y-1}{y}\right\} = (e^4\times 1) = e^4.$$

34. Given limit $= \lim\limits_{x\to 0}\dfrac{(e^{3x}-1)-(e^{2x}-1)}{x}$

$$= \lim_{x\to 0}\frac{(e^{3x}-1)}{x} - \lim_{x\to 0}\left\{\frac{e^{2x}-1}{x}\right\}$$

$$= \lim_{3x\to 0}\left\{3\left(\frac{e^{3x}-1}{3x}\right)\right\} - \lim_{2x\to 0}\left\{2\left(\frac{e^{2x}-1}{2x}\right)\right\}$$

$$= (3\times 1)-(2\times 1) = (3-2) = 1.$$

35. Given limit $= \lim\limits_{x\to 0}\left\{\left(\dfrac{e^x-1}{x}\right)-1\right\} = \lim\limits_{x\to 0}\left\{\dfrac{e^x-1}{x}\right\} - 1 = (1-1) = 0.$

36. Given limit $= \lim\limits_{x\to 0}\left\{\dfrac{(e^{bx}-1)-(e^{ax}-1)}{x}\right\} = \lim\limits_{bx\to 0}\left\{\left(\dfrac{e^{bx}-1}{bx}\right)\cdot b\right\} - \lim\limits_{ax\to 0}\left\{\left(\dfrac{e^{ax}-1}{ax}\right)\cdot a\right\}$

$$= (1\times b)-(1\times a) = (b-a).$$

37. Given limit $= \lim\limits_{x\to 0}\left\{\dfrac{(a^x-1)-(b^x-1)}{x}\right\} = \lim\limits_{x\to 0}\left\{\dfrac{a^x-1}{x}\right\} - \lim\limits_{x\to 0}\left\{\dfrac{b^x-1}{x}\right\} = (\log a - \log b).$

38. Given limit $= \lim\limits_{x\to 0}\left\{\left(\dfrac{a^{2x}-1}{2x}\right)\times\dfrac{2}{a^x}\right\} = \left[(\log a)\times\dfrac{2}{1}\right] = 2\log a.$

TRIGONOMETRIC LIMITS

For finding the limits of trigonometric functions, we use trigonometric transformations and simplify.

The following theorems are quite important.

THEOREM 1 *(i)* $\lim\limits_{\theta\to 0}\sin\theta = 0$ *(ii)* $\lim\limits_{\theta\to 0}\cos\theta = 1$

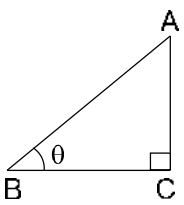
PROOF Consider a triangle ABC in which $\angle C$ is a right angle and $\angle ABC = \theta$ radians.

Then, $\sin\theta = \dfrac{CA}{BA}$ and $\cos\theta = \dfrac{BC}{BA}$.

If we keep BC fixed and go on decreasing θ, we find that A keeps on coming nearer and nearer to C.

$\therefore$ when $\theta\to 0$ then $A\to C$.

$\therefore$ $\theta\to 0 \Rightarrow CA\to 0$ and $BA\to BC$



$$\Rightarrow \frac{CA}{BA} \to 0 \text{ and } \frac{BC}{BA} \to 1$$
$$\Rightarrow \sin\theta \to 0 \text{ and } \cos\theta \to 1.$$

Thus, we have

(i) $\lim\limits_{\theta \to 0} \sin\theta = 0$, and (ii) $\lim\limits_{\theta \to 0} \cos\theta = 1.$

THEOREM 2 *(i)* $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ *(ii)* $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$

PROOF (i) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \lim\limits_{x \to 0} \dfrac{\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots\right)}{x}$ [by sine series]

$$= \lim_{x \to 0} \frac{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right)}{x} = \lim_{x \to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right) = 1.$$

$\therefore \quad \lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1.$

(ii) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = \lim\limits_{x \to 0}\left[\dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x}\right] = \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \dfrac{1}{\lim\limits_{x \to 0} \cos x} = (1 \times 1) = 1.$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$ *(ii)* $\lim\limits_{x \to 0} \dfrac{\sin 3x}{5x}$ *(iii)* $\lim\limits_{x \to 0}\left(\dfrac{\sin ax}{\sin bx}\right)$

SOLUTION (i) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x} = \lim\limits_{x \to 0}\left(\dfrac{2 \times \sin 2x}{2x}\right)$

$= 2 \cdot \lim\limits_{2x \to 0} \dfrac{\sin 2x}{2x}$ [$\because$ when $x \to 0$ then $2x \to 0$]

$= (2 \times 1) = 2$ $\left[\because \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1\right]$.

(ii) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{5x} = \lim\limits_{x \to 0}\left(\dfrac{3}{5} \cdot \dfrac{\sin 3x}{3x}\right)$

$= \dfrac{3}{5} \cdot \lim\limits_{3x \to 0} \dfrac{\sin 3x}{3x}$ [$\because$ when $x \to 0$ then $3x \to 0$]

$= \left(\dfrac{3}{5} \times 1\right) = \dfrac{3}{5}$ $\left[\because \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1\right]$.

$$\text{(iii)}\ \lim_{x\to 0}\left(\frac{\sin ax}{\sin bx}\right)=\lim_{x\to 0}\frac{ax\cdot\left(\frac{\sin ax}{ax}\right)}{bx\cdot\left(\frac{\sin bx}{bx}\right)}$$

$$=\frac{a}{b}\cdot\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin bx}{bx}\right)}$$

$$=\frac{a}{b}\cdot\frac{\lim\limits_{ax\to 0}\frac{\sin ax}{ax}}{\lim\limits_{bx\to 0}\frac{\sin bx}{bx}}\quad[\because\ x\to 0\ \Rightarrow\ ax\to 0\ \text{and}\ bx\to 0]$$

$$=\left(\frac{a}{b}\times\frac{1}{1}\right)=\frac{a}{b}\qquad\left[\because\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right].$$

EXAMPLE 2 *Evaluate* $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\tan 3x}$.

SOLUTION $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\tan 3x}=\lim\limits_{x\to 0}\dfrac{5x\cdot\left(\frac{\sin 5x}{5x}\right)}{3x\cdot\left(\frac{\tan 3x}{3x}\right)}$

$$=\frac{5}{3}\cdot\frac{\lim\limits_{5x\to 0}\left(\frac{\sin 5x}{5x}\right)}{\lim\limits_{3x\to 0}\left(\frac{\tan 3x}{3x}\right)}=\left(\frac{5}{3}\times\frac{1}{1}\right)=\frac{5}{3}.$$

EXAMPLE 3 *Evaluate* $\lim\limits_{x\to 0}\dfrac{(1-\cos x)}{x^2}$. **[CBSE 2000]**

SOLUTION $\lim\limits_{x\to 0}\dfrac{(1-\cos x)}{x^2}=\lim\limits_{x\to 0}\dfrac{2\sin^2(x/2)}{\left(\frac{x}{2}\right)^2\times 4}$

$$=\frac{1}{2}\lim_{x\to 0}\left\{\frac{\sin(x/2)}{(x/2)}\right\}^2=\left(\frac{1}{2}\times 1^2\right)=\frac{1}{2}.$$

EXAMPLE 4 *Evaluate (i)* $\lim\limits_{x\to 0}\left(\dfrac{1-\cos 4x}{1-\cos 5x}\right)$ **[CBSE 2002C]**

(ii) $\lim\limits_{x\to 0}\left(\dfrac{1-\cos mx}{1-\cos nx}\right)$.

SOLUTION (i) $\lim_{x\to 0}\left(\frac{1-\cos 4x}{1-\cos 5x}\right)=\lim_{x\to 0}\frac{2\sin^2 2x}{2\sin^2(5x/2)}$

$$=\lim_{x\to 0}\frac{\sin^2 2x}{\sin^2(5x/2)}=\lim_{x\to 0}\left\{\frac{\left\{\frac{\sin 2x}{2x}\right\}^2\cdot 4x^2}{\left\{\frac{\sin(5x/2)}{(5x/2)}\right\}^2\cdot\frac{25x^2}{4}}\right\}$$

$$=\frac{16}{25}\cdot\frac{\lim_{2x\to 0}\left(\frac{\sin 2x}{2x}\right)^2}{\lim_{\frac{5x}{2}\to 0}\left\{\frac{\sin(5x/2)}{(5x/2)}\right\}^2}=\left(\frac{16}{25}\times\frac{1^2}{1^2}\right)=\frac{16}{25}.$$

(ii) $\lim_{x\to 0}\left(\frac{1-\cos mx}{1-\cos nx}\right)$

$$=\lim_{x\to 0}\left\{\frac{2\sin^2(mx/2)}{2\sin^2(nx/2)}\right\}=\lim_{x\to 0}\left\{\frac{\sin^2(mx/2)}{\sin^2(nx/2)}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{\frac{\sin^2(mx/2)}{(mx/2)^2}\times\frac{m^2x^2}{4}}{\frac{\sin^2(nx/2)}{(nx/2)^2}\times\frac{n^2x^2}{4}}\right\}=\frac{m^2}{n^2}\cdot\frac{\lim_{\frac{mx}{2}\to 0}\left\{\frac{\sin(mx/2)}{(mx/2)}\right\}^2}{\lim_{\frac{nx}{2}\to 0}\left\{\frac{\sin(nx/2)}{(nx/2)}\right\}^2}$$

$$=\left(\frac{m^2}{n^2}\times\frac{1^2}{1^2}\right)=\frac{m^2}{n^2}.$$

EXAMPLE 5 *Evaluate* $\lim_{x\to 0}\left(\frac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}\right)$.

SOLUTION $\lim_{x\to 0}\left(\frac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}\right)$

$$=\lim_{x\to 0}\left(\frac{2\sin 4x\cos 2x}{2\cos 4x\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin 4x\cos 2x}{\cos 4x\sin x}\right)$$

$$=\lim_{x\to 0}\left\{\frac{\sin 4x}{4x}\times\frac{x}{\sin x}\times\cos 2x\times\frac{1}{\cos 4x}\times 4\right\}$$

$$=4\times\lim_{4x\to 0}\frac{\sin 4x}{4x}\times\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{2x\to 0}\cos 2x\times\frac{1}{\lim_{4x\to 0}\cos 4x}$$

$$=\left(4\times 1\times 1\times 1\times\frac{1}{1}\right)=4.$$

EXAMPLE 6 *Evaluate* $\lim_{x\to\pi}\left(\frac{1+\cos x}{\tan^2 x}\right)$.

SOLUTION $\lim_{x\to\pi}\left(\frac{1+\cos x}{\tan^2 x}\right)=\lim_{x\to\pi}\frac{\cos^2 x(1+\cos x)}{\sin^2 x}$

$$=\lim_{x\to\pi}\frac{\cos^2 x(1+\cos x)}{(1-\cos^2 x)}$$

$$=\lim_{x\to\pi}\frac{\cos^2 x}{(1-\cos x)}=\frac{1}{2} \qquad [\text{putting } x=\pi].$$

EXAMPLE 7 *Evaluate* $\lim_{x\to 0}\frac{x\tan x}{(1-\cos x)}$.

SOLUTION $\lim_{x\to 0}\frac{x\tan x}{(1-\cos x)}=\lim_{x\to 0}\frac{x\sin x}{\cos x(1-\cos x)}$

$$=\lim_{x\to 0}\frac{x\sin^2 x}{\sin x\cos x(1-\cos x)}$$

$$=\lim_{x\to 0}\frac{x(1-\cos^2 x)}{\sin x\cos x(1-\cos x)}=\lim_{x\to 0}\frac{x(1+\cos x)}{\sin x\cos x}$$

$$=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\frac{(1+\cos x)}{\cos x}=\left(1\times\frac{2}{1}\right)=2.$$

EXAMPLE 8 *Evaluate:*

(i) $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)$ *(ii)* $\lim_{x\to 0}\frac{(\operatorname{cosec} x-\cot x)}{x}$

SOLUTION (i) $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)=\lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)$

$$=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)}{\cos x}=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)\cos x}{\cos^2 x}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)\cos x}{(1-\sin^2 x)}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{(1+\sin x)}=0 \qquad \left[\text{putting } x=\frac{\pi}{2}\right].$$

$$\text{(ii)}\ \lim_{x\to 0}\frac{(\operatorname{cosec} x-\cot x)}{x}=\lim_{x\to 0}\frac{\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)}{x}$$

$$=\lim_{x\to 0}\frac{(1-\cos x)}{x\sin x}=\lim_{x\to 0}\frac{(1-\cos x)\ \sin x}{x\sin^2 x}$$

$$=\lim_{x\to 0}\frac{(1-\cos x)\ \sin x}{x\,(1-\cos^2 x)}=\lim_{x\to 0}\frac{\sin x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{(1+\cos x)}=\left(1\times\frac{1}{2}\right)=\frac{1}{2}\cdot$$

EXAMPLE 9 *Evaluate:* *(i)* $\lim_{x\to 0}\left(\frac{\tan x-\sin x}{\sin^3 x}\right)$ *(ii)* $\lim_{x\to 0}\frac{(\tan x-\sin x)}{x^3}$

SOLUTION (i) $\lim_{x\to 0}\frac{(\tan x-\sin x)}{\sin^3 x}=\lim_{x\to 0}\left(\frac{\sin x-\sin x\cos x}{\cos x\sin^3 x}\right)$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{\cos x\sin^3 x}=\lim_{x\to 0}\frac{(1-\cos x)}{\cos x\sin^2 x}$$

$$=\lim_{x\to 0}\frac{(1-\cos x)}{\cos x\,(1-\cos^2 x)}$$

$$=\lim_{x\to 0}\frac{1}{\cos x\,(1+\cos x)}=\frac{1}{2}\qquad[\text{putting } x=0].$$

$$\text{(ii)}\ \lim_{x\to 0}\frac{(\tan x-\sin x)}{x^3}=\lim_{x\to 0}\left(\frac{\sin x-\sin x\cos x}{x^3\cos x}\right)$$

$$=\lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{x^3\cos x}=\lim_{x\to 0}\left\{\frac{\sin x}{x}\cdot\frac{2\sin^2(x/2)}{x^2}\cdot\frac{1}{\cos x}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{\sin x}{x}\cdot\frac{2\sin^2(x/2)}{(x/2)^2}\cdot\frac{1}{4}\cdot\frac{1}{\cos x}\right\}$$

$$=\frac{1}{2}\cdot\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\left\{\frac{\sin(x/2)}{(x/2)}\right\}^2\cdot\lim_{x\to 0}\frac{1}{\cos x}$$

$$=\left(\frac{1}{2}\times 1\times 1^2\times 1\right)=\frac{1}{2}\cdot$$

EXAMPLE 10 *Evaluate* $\lim_{x\to 0}\left(\frac{x^3\cot x}{1-\cos x}\right)$.

SOLUTION $\lim_{x\to 0}\left(\frac{x^3\cot x}{1-\cos x}\right)$

$$=\lim_{x\to 0}\left\{\frac{x^3\cot x}{(1-\cos x)}\times\frac{(1+\cos x)}{(1+\cos x)}\right\}$$

$$=\lim_{x\to 0}\frac{x^3\cos x\,(1+\cos x)}{\sin x\,(1-\cos^2 x)}=\lim_{x\to 0}\frac{x^3\cos x\,(1+\cos x)}{\sin^3 x}$$

$$= \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^3 \times \lim_{x\to 0}\cos x \times \lim_{x\to 0}(1+\cos x)$$
$$= (1^3 \times 1 \times 2) = 2.$$

EXAMPLE 11 *Evaluate* $\lim_{x\to 0}\left(\frac{\sin 3x + 7x}{4x + \sin 2x}\right)$.

SOLUTION $\lim_{x\to 0}\left(\frac{\sin 3x + 7x}{4x + \sin 2x}\right) = \lim_{x\to 0}\left(\frac{\frac{\sin 3x}{x} + 7}{4 + \frac{\sin 2x}{x}}\right)$

[dividing num. and denom. by x]

$$= \lim_{x\to 0}\left[\frac{\left(\frac{\sin 3x}{3x}\right)\cdot 3 + 7}{4 + \left(\frac{\sin 2x}{2x}\right)\cdot 2}\right] = \frac{\left[3 \times \lim_{3x\to 0}\frac{\sin 3x}{3x}\right] + 7}{4 + \left[2 \times \lim_{2x\to 0}\left(\frac{\sin 2x}{2x}\right)\right]}$$
$$= \frac{(3\times 1) + 7}{4 + (2\times 1)} = \frac{10}{6} = \frac{5}{3}.$$

EXAMPLE 12 *Evaluate* $\lim_{x\to 0}\left(\frac{\tan 3x - 2x}{3x - \sin^2 x}\right)$.

SOLUTION $\lim_{x\to 0}\left(\frac{\tan 3x - 2x}{3x - \sin^2 x}\right)$

$$= \lim_{x\to 0}\frac{\left(\frac{\tan 3x}{3x} - \frac{2}{3}\right)}{\left(1 - \frac{1}{3}\cdot\frac{\sin x}{x}\cdot\sin x\right)}$$ [dividing num. and denom. by $3x$]

$$= \frac{\left(\lim_{3x\to 0}\frac{\tan 3x}{3x} - \frac{2}{3}\right)}{\left\{1 - \frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\sin x\right\}} = \frac{\left(1 - \frac{2}{3}\right)}{1 - \left(\frac{1}{3}\times 1\times 0\right)} = \frac{\left(\frac{1}{3}\right)}{1} = \frac{1}{3}.$$

EXAMPLE 13 *Evaluate* $\lim_{x\to 0}\frac{x\tan 4x}{1-\cos 4x}$. **[CBSE 2004]**

SOLUTION $\lim_{x\to 0}\frac{x\tan 4x}{1-\cos 4x} = \lim_{x\to 0}\frac{x\sin 4x}{\cos 4x\,(2\sin^2 2x)}$

$$= \lim_{x\to 0}\frac{2x\sin 2x\cos 2x}{\cos 4x\,(2\sin^2 2x)} = \lim_{x\to 0}\left(\frac{\cos 2x}{\cos 4x}\cdot\frac{2x}{\sin 2x}\times\frac{1}{2}\right)$$
$$= \frac{1}{2}\times\frac{\lim_{2x\to 0}\cos 2x}{\lim_{4x\to 0}\cos 4x}\times\lim_{2x\to 0}\left(\frac{2x}{\sin 2x}\right) = \left(\frac{1}{2}\times\frac{1}{1}\times 1\right) = \frac{1}{2}.$$

EXAMPLE 14 *Evaluate* $\lim_{x\to 0}\dfrac{(1-\cos x\sqrt{\cos 2x})}{x^2}$. **[CBSE 2004C, '05]**

SOLUTION $\lim_{x\to 0}\dfrac{(1-\cos x\sqrt{\cos 2x})}{x^2}$

$$=\lim_{x\to 0}\left\{\frac{(1-\cos x\sqrt{\cos 2x})}{x^2}\times\frac{(1+\cos x\sqrt{\cos 2x})}{(1+\cos x\sqrt{\cos 2x})}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{(1-\cos^2 x\cos 2x)}{x^2}\times\frac{1}{(1+\cos x\sqrt{\cos 2x})}\right\}$$

$$=\lim_{x\to 0}\frac{(1-\cos^2 x\cos 2x)}{x^2}\times\lim_{x\to 0}\frac{1}{(1+\cos x\sqrt{\cos 2x})}$$

$$=\lim_{x\to 0}\left\{\frac{1-\cos^2 x(1-2\sin^2 x)}{x^2}\right\}\times\frac{1}{(1+1\sqrt{1})}$$

$$=\lim_{x\to 0}\frac{(1-\cos^2 x+2\sin^2 x\cos^2 x)}{x^2}\times\frac{1}{2}$$

$$=\frac{1}{2}\times\lim_{x\to 0}\frac{(\sin^2 x+2\sin^2 x\cos^2 x)}{x^2}=\frac{1}{2}\times\lim_{x\to 0}\frac{\sin^2 x(1+2\cos^2 x)}{x^2}$$

$$=\frac{1}{2}\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\times\lim_{x\to 0}(1+2\cos^2 x)$$

$$=\frac{1}{2}\times 1^2\times(1+2\times 1^2)=\frac{3}{2}.$$

EXAMPLE 15 *Evaluate* $\lim_{x\to\frac{\pi}{4}}\dfrac{(\sin x-\cos x)}{\left(x-\frac{\pi}{4}\right)}$. **[CBSE 2000, '03]**

SOLUTION Put $\left(x-\frac{\pi}{4}\right)=y$ so that when $x\to\frac{\pi}{4}$ then $y\to 0$.

$$\therefore\quad\lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)}{\left(x-\frac{\pi}{4}\right)}$$

$$=\lim_{y\to 0}\frac{\left\{\sin\left(\frac{\pi}{4}+y\right)-\cos\left(\frac{\pi}{4}+y\right)\right\}}{y}\qquad[\text{putting}\,(x-\tfrac{\pi}{4})=y]$$

$$=\lim_{y\to 0}\frac{\left\{\left(\sin\frac{\pi}{4}\cos y+\cos\frac{\pi}{4}\sin y\right)-\left(\cos\frac{\pi}{4}\cos y-\sin\frac{\pi}{4}\sin y\right)\right\}}{y}$$

$$=\frac{2}{\sqrt{2}}\times\lim_{y\to 0}\left(\frac{\sin y}{y}\right)=(\sqrt{2}\times 1)=\sqrt{2}.$$

EXAMPLE 16 *Evaluate* $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\cos x}{\left(\frac{\pi}{2}-x\right)}$.

SOLUTION Put $\left(x-\dfrac{\pi}{2}\right)=y$, so that when $x\to\dfrac{\pi}{2}$ then $y\to 0$.

$$\therefore \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\left(\frac{\pi}{2}-x\right)} = \lim_{y\to 0} \frac{\cos\left(\frac{\pi}{2}+y\right)}{-y} = \lim_{y\to 0}\left(\frac{-\sin y}{-y}\right) = \lim_{y\to 0}\frac{\sin y}{y} = 1.$$

EXAMPLE 17 *Evaluate* $\lim\limits_{x\to\frac{\pi}{6}} \dfrac{(\sqrt{3}\sin x-\cos x)}{\left(x-\frac{\pi}{6}\right)}$.

SOLUTION $\lim\limits_{x\to\frac{\pi}{6}} \dfrac{(\sqrt{3}\sin x-\cos x)}{\left(x-\frac{\pi}{6}\right)}$

$$= \lim_{x\to\frac{\pi}{6}} \frac{2\left(\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x\right)}{\left(x-\frac{\pi}{6}\right)} = \lim_{x\to\frac{\pi}{6}} \frac{2\left(\sin x\cos\frac{\pi}{6}-\cos x\sin\frac{\pi}{6}\right)}{\left(x-\frac{\pi}{6}\right)}$$

$$= 2\lim_{x\to\frac{\pi}{6}} \frac{\sin\left(x-\frac{\pi}{6}\right)}{\left(x-\frac{\pi}{6}\right)} = 2\lim_{y\to 0}\frac{\sin y}{y} = 2 \qquad [\text{putting } \left(x-\tfrac{\pi}{6}\right)=y].$$

EXERCISE 27B

Evaluate the following limits:

1. $\lim\limits_{x\to 0}\dfrac{\sin 4x}{6x}$

2. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\sin 8x}$

3. $\lim\limits_{x\to 0}\dfrac{\tan 3x}{\tan 5x}$

4. $\lim\limits_{x\to 0}\dfrac{\tan \alpha x}{\tan \beta x}$

5. $\lim\limits_{x\to 0}\dfrac{\sin 4x}{\tan 7x}$

6. $\lim\limits_{x\to 0}\dfrac{\tan 3x}{\sin 4x}$

7. $\lim\limits_{x\to 0}\dfrac{\sin mx}{\tan nx}$

8. $\lim\limits_{x\to 0}\dfrac{(\sin x-2\sin 3x+\sin 5x)}{x}$

9. $\lim\limits_{x\to \pi/6}\dfrac{(2\sin^2 x+\sin x-1)}{(2\sin^2 x-3\sin x+1)}$

10. $\lim\limits_{x\to 0}\dfrac{(\sin 2x+3x)}{(2x+\sin 3x)}$

11. $\lim_{x\to 0} \dfrac{(\tan 2x - x)}{(3x - \tan x)}$

12. $\lim_{x\to 0} \dfrac{(x^2 - \tan 2x)}{\tan x}$ **[CBSE 1998]**

13. $\lim_{x\to 0} \dfrac{(x\cos x + \sin x)}{(x^2 + \tan x)}$

14. $\lim_{x\to 0} \dfrac{(\tan x - \sin x)}{\sin^3 x}$

15. $\lim_{x\to 0} (x\operatorname{cosec} x)$

16. $\lim_{x\to 0} (x\cot 2x)$

17. $\lim_{x\to 0} \dfrac{\sin x \cos x}{3x}$

18. $\lim_{x\to 0} \dfrac{\sin (x/4)}{x}$

19. $\lim_{x\to 0} \dfrac{\tan (x/2)}{3x}$

20. $\lim_{x\to 0} \dfrac{(1 - \cos x)}{\sin^2 x}$ **[CBSE 2000]**

21. $\lim_{x\to 0} \dfrac{(1 - \cos 3x)}{x^2}$ **[CBSE 2000]**

22. $\lim_{x\to 0} \dfrac{(1 - \cos 2x)}{\sin^2 2x}$

23. $\lim_{x\to 0} \dfrac{(1 - \cos 2x)}{3\tan^2 x}$

24. $\lim_{x\to 0} \dfrac{(1 - \cos 4x)}{(1 - \cos 6x)}$ **[CBSE 2002C]**

25. $\lim_{x\to 0} \dfrac{(1 - \cos mx)}{(1 - \cos nx)}$

26. $\lim_{x\to 0} \dfrac{(2\sin x - \sin 2x)}{x^3}$

27. $\lim_{x\to 0} \dfrac{(\tan x - \sin x)}{x^3}$

28. $\lim_{x\to 0} \dfrac{(\tan 2x - \sin 2x)}{x^3}$

29. $\lim_{x\to 0} \dfrac{(\operatorname{cosec} x - \cot x)}{x}$

30. $\lim_{x\to 0} \dfrac{(\cot 2x - \operatorname{cosec} 2x)}{x}$

31. $\lim_{x\to 0} \dfrac{\sin 2x(1 - \cos 2x)}{x^3}$

32. $\lim_{x\to \frac{\pi}{4}} \dfrac{(\sec^2 x - 2)}{(\tan x - 1)}$ **[CBSE 2001, '2C]**

33. $\lim_{x\to \frac{\pi}{4}} \dfrac{(\operatorname{cosec}^2 x - 2)}{(\cot x - 1)}$ **[CBSE 2001]**

34. $\lim_{x\to \frac{\pi}{4}} \dfrac{(1 - \tan x)}{\left(x - \frac{\pi}{4}\right)}$ **[CBSE 2003C]**

35. $\lim_{x\to \pi} \dfrac{(\sin 3x - 3\sin x)}{(\pi - x)^3}$

36. $\lim_{x\to \frac{\pi}{2}} \dfrac{(1 + \cos 2x)}{(\pi - 2x)^2}$ **[CBSE 2003C]**

37. $\lim_{x\to a} \dfrac{(\cos x - \cos a)}{(x - a)}$

38. $\lim_{x\to a} \dfrac{(\sin x - \sin a)}{(x - a)}$

39. $\lim_{x\to a} \dfrac{(\sin x - \sin a)}{(\sqrt{x} - \sqrt{a})}$

40. $\lim_{x\to 0} \dfrac{(\sin 5x - \sin 3x)}{\sin x}$

41. $\lim_{x\to 0} \dfrac{(\cos 3x - \cos 5x)}{x^2}$

42. $\lim_{x\to 0} \dfrac{(\sin 3x + \sin 5x)}{(\sin 6x - \sin 4x)}$

43. $\lim_{x\to 0} \dfrac{[\sin (2 + x) - \sin (2 - x)]}{x}$

44. $\lim_{x\to 0} \dfrac{(1 - \cos 2x)}{(\cos 2x - \cos 8x)}$

45. $\lim\limits_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x$

46. $\lim\limits_{x \to 0} \frac{(\sqrt{1+2x} - \sqrt{1-2x})}{\sin x}$ **[CBSE 1998C]**

47. $\lim\limits_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

48. $\lim\limits_{x \to 0} \frac{(e^{3+x} - \sin x - e^3)}{x}$

49. $\lim\limits_{x \to 0} \frac{(e^{\tan x} - 1)}{\tan x}$

50. $\lim\limits_{x \to 0} \frac{(e^{\tan x} - 1)}{x}$

51. $\lim\limits_{x \to 0} \frac{ax + x\cos x}{b \sin x}$

52. $\lim\limits_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$, where $a, b, a + b \neq 0$

53. $\lim\limits_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

54. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$

55. $\lim\limits_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

56. $\lim\limits_{x \to 0} (\operatorname{cosec} x - \cot x)$

57. $\lim\limits_{x \to 0} \frac{1 - \cos 2mx}{1 - \cos 2nx}$

58. $\lim\limits_{x \to 0} \frac{1 - \cos mx}{1 - \cos nx}$

59. $\lim\limits_{x \to 0} \frac{\sin^2 mx}{\sin^2 nx}$

60. $\lim\limits_{x \to 0} \frac{\sin 2x + \sin 3x}{2x + \sin 3x}$

61. $\lim\limits_{x \to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x}$

62. $\lim\limits_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$

63. $\lim\limits_{x \to 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x}$

64. $\lim\limits_{x \to \frac{\pi}{6}} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2}$

65. $\lim\limits_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$

66. $\lim\limits_{x \to a} \frac{\cos x - \cos a}{\cot x - \cot a}$

67. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$

68. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\sqrt{2}\cos^2 x}$

69. $\lim\limits_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

70. $\lim\limits_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$

71. $\lim\limits_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2}\sin x}$

72. $\lim\limits_{x \to \frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$

ANSWERS (EXERCISE 27B)

1. $\frac{2}{3}$ **2.** $\frac{5}{8}$ **3.** $\frac{3}{5}$ **4.** $\frac{\alpha}{\beta}$ **5.** $\frac{4}{7}$ **6.** $\frac{3}{4}$ **7.** $\frac{m}{n}$ **8.** 0

9. –3 **10.** 1 **11.** $\frac{1}{2}$ **12.** –2 **13.** 2 **14.** $\frac{1}{2}$ **15.** 1 **16.** $\frac{1}{2}$

17. $\frac{1}{3}$ **18.** $\frac{1}{4}$ **19.** $\frac{1}{6}$ **20.** $\frac{1}{2}$ **21.** $\frac{9}{2}$ **22.** $\frac{1}{2}$ **23.** $\frac{2}{3}$ **24.** $\frac{4}{9}$

25. $\frac{m^2}{n^2}$ **26.** 1 **27.** $\frac{1}{2}$ **28.** 4 **29.** $\frac{1}{2}$ **30.** -1 **31.** 4 **32.** 2

33. 2 **34.** -2 **35.** -1 **36.** $\frac{1}{2}$ **37.** $-\sin a$ **38.** $\cos a$

39. $2\sqrt{a}\cos a$ **40.** 2 **41.** 8 **42.** 4 **43.** $2\cos 2$ **44.** $\frac{1}{15}$

45. 1 **46.** 2 **47.** $(2a\sin a + a^2\cos a)$ **48.** $(e^3 - 1)$ **49.** 1

50. 1 **51.** $\frac{a+1}{b}$ **52.** 1 **53.** $\frac{1}{\pi}$ **54.** 2 **55.** 4 **56.** 0 **57.** $\frac{m^2}{n^2}$

58. $\frac{m}{n}$ **59.** $\frac{m^2}{n^2}$ **60.** 1 **61.** $\frac{3}{2}$ **62.** $\frac{1}{4\sqrt{2}}$ **63.** 1 **64.** $\frac{1}{36}$

65. $\frac{(a^2-b^2)}{c^2}$ **66.** $\sin^3 a$ **67.** -4 **68.** $\frac{1}{8}$ **69.** 4 **70.** $\frac{1}{4}$

71. 2 **72.** -3

HINTS TO SOME SELECTED QUESTIONS

8. Given limit $= \left(\lim_{x\to 0}\frac{\sin x}{x}\right) - (2\times 3)\cdot\left(\lim_{x\to 0}\frac{\sin 3x}{3x}\right) + 5\cdot\left(\lim_{x\to 0}\frac{\sin 5x}{5x}\right)$

$= [1 - (6\times 1) + (5\times 1)] = 0.$

9. Given limit $= \lim_{x\to\frac{\pi}{6}}\frac{(2\sin x - 1)(\sin x + 1)}{(2\sin x - 1)(\sin x - 1)} = \lim_{x\to\frac{\pi}{6}}\frac{(\sin x + 1)}{(\sin x - 1)}$

$= \frac{\left(\sin\frac{\pi}{6} + 1\right)}{\left(\sin\frac{\pi}{6} - 1\right)} = -3.$

10. Given limit $= \frac{2\times\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right) + 3}{2 + \left(3\times\lim_{x\to 0}\frac{\sin 3x}{3x}\right)} = \frac{(2\times 1) + 3}{2 + (3\times 1)} = \frac{5}{5} = 1.$

11. Given limit $= \frac{2\times\left(\lim_{x\to 0}\frac{\tan 2x}{2x}\right) - 1}{3 - \lim_{x\to 0}\left(\frac{\tan x}{x}\right)} = \frac{(2\times 1) - 1}{(3 - 1)} = \frac{1}{2}.$

12. Given limit $= \lim_{x\to 0}\frac{\left(x - 2\times\frac{\tan 2x}{2x}\right)}{\left(\frac{\tan x}{x}\right)} = \left\{\frac{0 - (2\times 1)}{1}\right\} = -2.$

13. Given limit $= \lim_{x\to 0}\frac{\left(\cos x + \frac{\sin x}{x}\right)}{\left(x + \frac{\tan x}{x}\right)} = \frac{(1+1)}{(0+1)} = 2.$

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14. Given limit $= \lim_{x\to 0} \dfrac{\sin x(1-\cos x)}{\cos x \sin^3 x} = \lim_{x\to 0} \dfrac{(1-\cos x)}{\cos x \sin^2 x}$

$= \lim_{x\to 0} \dfrac{(1-\cos x)}{\cos x(1-\cos^2 x)} = \lim_{x\to 0} \dfrac{1}{\cos x(1+\cos x)} = \dfrac{1}{2}.$

15. Given limit $= \lim_{x\to 0} \dfrac{x}{\sin x} = 1.$

16. Given limit $= \dfrac{1}{2} \times \lim_{2x\to 0} \dfrac{2x}{\tan 2x} = \left(\dfrac{1}{2}\times 1\right) = \dfrac{1}{2}.$

20. Given limit $= \lim_{x\to 0} \dfrac{(1-\cos x)}{(1-\cos^2 x)} = \lim_{x\to 0} \dfrac{1}{(1+\cos x)} = \dfrac{1}{2}.$

21. Given limit $= \lim_{x\to 0} \dfrac{2\sin^2\left(\dfrac{3x}{2}\right)}{\left(\dfrac{3x}{2}\right)^2 \times \dfrac{4}{9}} = \dfrac{9}{2}.$

24. Given limit $= \lim_{x\to 0} \dfrac{2\sin^2 2x}{2\sin^2 3x} = \lim_{x\to 0} \dfrac{\sin^2 2x}{\sin^2 3x}$

$= \lim_{x\to 0} \dfrac{\left(\dfrac{\sin 2x}{2x}\right)^2 \times 4x^2}{\left(\dfrac{\sin 3x}{3x}\right)^2 \times 9x^2} = \dfrac{4}{9} \cdot \dfrac{\lim\limits_{2x\to 0}\left(\dfrac{\sin 2x}{2x}\right)^2}{\lim\limits_{3x\to 0}\left(\dfrac{\sin 3x}{3x}\right)^2}$

$= \left(\dfrac{4}{9}\times\dfrac{1}{1}\right) = \dfrac{4}{9}.$

26. Given limit $= \lim_{x\to 0} \dfrac{2\sin x(1-\cos x)}{x^3} = \lim_{x\to 0} \left[2\left(\dfrac{\sin x}{x}\right) \cdot \dfrac{2\sin^2(x/2)}{\left(\dfrac{x}{2}\right)^2 \times 4}\right] = 1.$

29. Given limit $= \lim_{x\to 0} \dfrac{(1-\cos x)}{x\sin x} = \lim_{x\to 0} \dfrac{2\sin^2(x/2)}{2x\sin(x/2)\cos(x/2)} = \lim_{x\to 0} \dfrac{\tan(x/2)}{\left(\dfrac{x}{2}\right)} \times \dfrac{1}{2} = \dfrac{1}{2}.$

30. Given limit $= \lim_{x\to 0} \dfrac{(\cos 2x - 1)}{x\sin 2x} = \lim_{x\to 0} \dfrac{-(1-\cos 2x)}{x\sin 2x}$

$= \lim_{x\to 0} \dfrac{-2\sin^2 x}{2x\sin x\cos x} = -\lim_{x\to 0} \dfrac{\tan x}{x} = -1.$

35. Given limit $= \lim_{x\to \pi} \dfrac{(3\sin x - 4\sin^3 x - 3\sin x)}{(\pi - x)^3}$

$= \lim_{x\to \pi} \dfrac{-4\sin^3 x}{(\pi - x)^3} = \lim_{\theta\to 0} \dfrac{-4\sin^3(\pi+\theta)}{(-\theta)^3}$, where $(x-\pi) = \theta$

$= -4 \lim_{\theta\to 0} \left(\dfrac{\sin\theta}{\theta}\right)^3 = (-4\times 1) = -4.$

36. Putting $(\pi - 2x) = t$, we have

$$\text{given limit} = \lim_{t\to 0}\frac{1+\cos(\pi - t)}{t^2} = \lim_{t\to 0}\frac{1-\cos t}{t^2}$$

$$= \lim_{t\to 0}\frac{2\sin^2(t/2)}{t^2} = \frac{2}{4}\cdot\lim_{t\to 0}\left\{\frac{\sin(t/2)}{(t/2)}\right\}^2 = \left(\frac{1}{2}\times 1^2\right) = \frac{1}{2}.$$

37. Given limit $= \lim_{x\to a}\dfrac{-2\sin\left(\dfrac{x+a}{2}\right)\sin\left(\dfrac{x-a}{2}\right)}{(x-a)} = -\lim_{x\to a}\left\{\sin\left(\dfrac{x+a}{2}\right)\cdot\dfrac{\sin\left(\dfrac{x-a}{2}\right)}{\left(\dfrac{x-a}{2}\right)}\right\}$

$$= -\sin a\times 1 = -\sin a.$$

39. Given limit $= \lim_{x\to a}\dfrac{2\cos\left(\dfrac{x+a}{2}\right)\sin\left(\dfrac{x-a}{2}\right)}{(x-a)}\times(\sqrt{x}+\sqrt{a})$

$$= \lim_{x\to a}\left\{\cos\left(\frac{x+a}{2}\right)\cdot\frac{\sin\left(\dfrac{x-a}{2}\right)}{\left(\dfrac{x-a}{2}\right)}\times(\sqrt{x}+\sqrt{a})\right\}$$

$$= (\cos a\times 1\times 2\sqrt{a}) = 2\sqrt{a}\cos a.$$

41. Given limit $= \lim_{x\to 0}\dfrac{2\sin 4x\sin x}{x^2} = \lim_{x\to 0}\left(\dfrac{\sin 4x}{4x}\times\dfrac{\sin x}{x}\times 8\right) = (8\times 1\times 1) = 8.$

42. Given limit $= \lim_{x\to 0}\dfrac{2\sin 4x\cos x}{2\cos 5x\sin x}$

$$= \lim_{x\to 0}\left\{\frac{\left(\dfrac{\sin 4x}{4x}\times 4\right)\cdot\cos x}{\left(\dfrac{\sin x}{x}\right)\cdot\cos 5x}\right\} = \frac{(1\times 4)\times 1}{1\times 1} = 4.$$

43. Given limit $= \lim_{x\to 0}\dfrac{2\cos\left\{\dfrac{(2+x+2-x)}{2}\right\}\sin\left\{\dfrac{(2+x-2+x)}{2}\right\}}{x}$

$$= (2\cos 2)\cdot\lim_{x\to 0}\frac{\sin x}{x} = (2\cos 2)\times 1 = (2\cos 2).$$

44. Given limit $= \lim_{x\to 0}\dfrac{2\sin^2 x}{2\sin 5x\sin 3x} = \lim_{x\to 0}\dfrac{\sin x\cdot\sin x}{\sin 5x\cdot\sin 3x}$

$$= \frac{\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)\cdot\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)}{\lim_{x\to 0}\left(\dfrac{\sin 5x}{5x}\times 5\right)\times\lim_{x\to 0}\left(\dfrac{\sin 3x}{3x}\times 3\right)}$$

$$= \frac{1\times 1}{(1\times 5)\times(1\times 3)} = \frac{1}{15}.$$

45. Put $\left(\frac{\pi}{2} - x\right) = y$ so that when $x \to \frac{\pi}{2}$ then $y \to 0$.

$\therefore$ given limit $= \lim_{y \to 0} y \tan\left(\frac{\pi}{2} - y\right) = \lim_{y \to 0} y \cot y = \lim_{y \to 0} \frac{y}{\tan y} = 1.$

46. Given limit $= \lim_{x \to 0} \frac{(\sqrt{1+2x} - \sqrt{1-2x})(\sqrt{1+2x} + \sqrt{1-2x})}{\sin x(\sqrt{1+2x} + \sqrt{1-2x})}$

$= \lim_{x \to 0} \frac{\{(1+2x) - (1-2x)\}}{\sin x(\sqrt{1+2x} + \sqrt{1-2x})}$

$= 4 \cdot \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{1}{(\sqrt{1+2x} + \sqrt{1-2x})}$

$= \left(4 \times 1 \times \frac{1}{2}\right) = 2.$

47. Given limit $= \lim_{h \to 0} \frac{(a^2 + h^2 + 2ah)\sin(a+h) - a^2 \sin a}{h}$

$= a^2 \cdot \lim_{h \to 0} \frac{\sin(a+h) - \sin a}{h} + \lim_{h \to 0} h \sin(a+h) + \lim_{h \to 0} 2a \sin(a+h)$

$= a^2 \cdot \lim_{h \to 0} \frac{\cos\left(a + \frac{h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{(h/2)} + 0 + 2a \sin a$

$= a^2 \cdot \lim_{h \to 0} \cos\left(a + \frac{h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin(h/2)}{(h/2)} + 2a \sin a$

$= (a^2 \times \cos a \times 1) + 2a \sin a = (a^2 \cos a + 2a \sin a).$

48. Given limit $= e^3 \cdot \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) - \lim_{x \to 0} \frac{\sin x}{x} = (e^3 \times 1 - 1) = (e^3 - 1).$

50. Given limit $= \lim_{x \to 0} \frac{(e^{\tan x} - 1)}{x} = \lim_{x \to 0} \left\{\frac{(e^{\tan x} - 1)}{\tan x} \times \frac{\tan x}{x}\right\}$

$= \lim_{\tan x \to 0} \frac{(e^{\tan x} - 1)}{\tan x} \times \lim_{x \to 0} \frac{\tan x}{x} = (1 \times 1) = 1.$

51. $\lim_{x \to 0} \frac{ax + x\cos x}{b \sin x} = \lim_{x \to 0} \frac{a + \cos x}{b\left(\frac{\sin x}{x}\right)}$ [dividing num. and denom. by x]

$= \frac{(a+1)}{(b \times 1)} = \frac{(a+1)}{b}.$

52. $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx} = \lim_{x \to 0} \left\{\frac{\frac{\sin ax}{x} + b}{a + \frac{\sin bx}{x}}\right\}$ [dividing num. and denom. by x]

$= \lim_{x \to 0} \frac{\left\{a\left(\frac{\sin ax}{ax}\right) + b\right\}}{\left\{a + b\left(\frac{\sin bx}{bx}\right)\right\}} = \frac{(a \times 1) + b}{a + (b \times 1)} = \frac{a+b}{a+b} = 1.$

53. Putting $(\pi - x) = h$, we have $x \to \pi \Rightarrow \pi - x \to 0 \Rightarrow h \to 0$.

$$\therefore \lim_{x\to\pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \frac{1}{\pi}\lim_{h\to 0}\frac{\sin h}{h} = \left(\frac{1}{\pi}\times 1\right) = \frac{1}{\pi}.$$

54. Put $x - \frac{\pi}{2} = h$. Then, $x \to \frac{\pi}{2} \Rightarrow \left(x - \frac{\pi}{2}\right) \to 0 \Rightarrow h \to 0$.

$$\therefore \lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{\left(x - \frac{\pi}{2}\right)} = \lim_{h\to 0}\frac{\tan 2\left(h + \frac{\pi}{2}\right)}{h} = 2\lim_{h\to 0}\frac{\tan(2h + \pi)}{2h}$$

$$= 2\lim_{h\to 0}\frac{\tan 2h}{2h} = (2\times 1) = 2.$$

55. $\lim_{x\to 0}\frac{\cos 2x - 1}{\cos x - 1} = \lim_{x\to 0}\frac{2\cos^2 x - 2}{\cos x - 1} = \lim_{x\to 0}\frac{2(\cos^2 x - 1)}{(\cos x - 1)} = 2\lim_{x\to 0}(\cos x + 1) = 4.$

56. Given limit $= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\frac{(1 - \cos x)}{\sin x} = \lim_{x\to 0}\frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}$

$$= \lim_{x\to 0}\tan\left(\frac{x}{2}\right) = 0.$$

57. Given limit $= \lim_{x\to 0}\frac{2\sin^2 mx}{2\sin^2 nx} = \lim_{x\to 0}\frac{\sin^2 mx}{\sin^2 nx} = \lim_{x\to 0}\left\{\frac{\frac{\sin mx}{mx}\times mx}{\frac{\sin nx}{nx}\times nx}\times\frac{\frac{\sin mx}{mx}\times mx}{\frac{\sin nx}{nx}\times nx}\right\}$

$$= \frac{m^2}{n^2}\cdot\left\{\frac{\lim_{x\to 0}\frac{\sin mx}{mx}}{\lim_{x\to 0}\frac{\sin nx}{nx}}\right\}\times\left\{\frac{\lim_{x\to 0}\frac{\sin mx}{mx}}{\lim_{x\to 0}\frac{\sin nx}{nx}}\right\} = \left(\frac{m^2}{n^2}\times\frac{1}{1}\times\frac{1}{1}\right) = \frac{m^2}{n^2}.$$

58. Given limit $= \lim_{x\to 0}\frac{2\sin^2\left(\frac{mx}{2}\right)}{2\sin^2\left(\frac{nx}{2}\right)} = \lim_{x\to 0}\left\{\frac{\sin\frac{mx}{2}}{\sin\frac{nx}{2}}\right\}^2$

$$= \lim_{x\to 0}\left\{\frac{\frac{\sin\frac{mx}{2}}{\left(\frac{mx}{2}\right)}\times\frac{mx}{2}}{\frac{\sin\frac{nx}{x}}{\left(\frac{nx}{2}\right)}\times\left(\frac{nx}{2}\right)}\right\}^2 = \frac{m}{n}\cdot\left\{\frac{\lim_{x\to 0}\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}}{\left\{\lim_{x\to 0}\frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right\}}\right\}$$

$$= \left(\frac{m}{n}\times\frac{1}{1}\right) = \frac{m}{n}.$$

59. $\lim_{x\to 0}\frac{\sin^2 mx}{\sin^2 nx} = \lim_{x\to 0}\left\{\frac{\frac{\sin mx}{mx}\times mx\cdot\frac{\sin mx}{mx}\times mx}{\frac{\sin nx}{nx}\times nx\cdot\frac{\sin nx}{nx}\times nx}\right\}$

$$= \frac{m^2}{n^2} \cdot \frac{\lim\limits_{x\to 0}\left(\frac{\sin mx}{mx} \cdot \frac{\sin mx}{mx}\right)}{\lim\limits_{x\to 0}\left(\frac{\sin nx}{nx} \cdot \frac{\sin nx}{nx}\right)} = \frac{m^2}{n^2} \cdot \frac{\lim\limits_{x\to 0}\frac{\sin mx}{mx} \cdot \lim\limits_{x\to 0}\frac{\sin mx}{mx}}{\lim\limits_{x\to 0}\frac{\sin nx}{nx} \cdot \lim\limits_{x\to 0}\frac{\sin nx}{nx}}$$

$$= \left(\frac{m^2}{n^2} \times \frac{1\times 1}{1\times 1}\right) = \frac{m^2}{n^2}.$$

60. $\lim\limits_{x\to 0} \dfrac{\sin 2x + \sin 3x}{2x + \sin 3x} = \lim\limits_{x\to 0} \left\{\dfrac{\frac{\sin 2x}{x} + \frac{\sin 3x}{x}}{2 + \frac{\sin 3x}{x}}\right\}$ [dividing num. and denom. by x]

$$= \lim_{x\to 0} \frac{2\times \frac{\sin 2x}{2x} + 3\times \frac{\sin 3x}{3x}}{2 + 3\times \frac{\sin 3x}{3x}}$$

$$= \frac{2\times \lim\limits_{x\to 0} \frac{\sin 2x}{2x} + 3\times \lim\limits_{x\to 0} \frac{\sin 3x}{3x}}{2 + 3\times \lim\limits_{x\to 0} \frac{\sin 3x}{3x}} = \frac{(2\times 1 + 3\times 1)}{(2 + 3\times 1)} = \frac{5}{5} = 1.$$

61. $\lim\limits_{x\to 0} \dfrac{\sec 4x - \sec 2x}{\sec 3x - \sec x} = \lim\limits_{x\to 0} \dfrac{\left\{\frac{1}{\cos 4x} - \frac{1}{\cos 2x}\right\}}{\left\{\frac{1}{\cos 3x} - \frac{1}{\cos x}\right\}} = \lim\limits_{x\to 0} \left\{\dfrac{\frac{\cos 2x - \cos 4x}{\cos 4x \cos 2x}}{\frac{\cos x - \cos 3x}{\cos 3x \cos x}}\right\}$

$$= \lim_{x\to 0} \left\{\frac{\cos 2x - \cos 4x}{\cos x - \cos 3x} \times \frac{\cos 3x \cos x}{\cos 4x \cos 2x}\right\}$$

$$= \lim_{x\to 0} \left\{\frac{2\sin 3x \sin x}{2\sin 2x \sin x} \times \frac{\cos x \cos 3x}{\cos 2x \cos 4x}\right\}$$

$$= \lim_{x\to 0} \left\{\frac{\frac{\sin 3x}{3x}\times 3}{\frac{\sin 2x}{2x}\times 2}\right\} \times \lim_{x\to 0} \frac{\cos x \cos 3x}{\cos 2x \cos 4x}$$

$$= \left(\frac{1\times 3}{1\times 2} \times \frac{1}{1}\right) = \frac{3}{2}.$$

62. $\lim\limits_{x\to 0} \dfrac{\sqrt{2} - \sqrt{1+\cos x}}{\sin^2 x} = \lim\limits_{x\to 0} \dfrac{\sqrt{2} - \sqrt{2\cos^2(x/2)}}{(1-\cos^2 x)} = \lim\limits_{x\to 0} \dfrac{\sqrt{2}\left(1 - \cos\frac{x}{2}\right)}{(1-\cos x)(1+\cos x)}$

$$= \lim_{x\to 0} \frac{\sqrt{2}\left(1 - \cos\frac{x}{2}\right)}{\left(2\sin^2\frac{x}{2}\right)(1+\cos x)} = \frac{1}{\sqrt{2}} \lim_{x\to 0} \frac{\left(1 - \cos\frac{x}{2}\right)}{\left(1 - \cos^2\frac{x}{2}\right)(1+\cos x)}$$

$$= \frac{1}{\sqrt{2}} \lim_{x\to 0} \frac{1}{\left(1 + \cos\frac{x}{2}\right)(1+\cos x)} = \frac{1}{\sqrt{2}} \times \frac{1}{(1+1)(1+1)} = \frac{1}{4\sqrt{2}}.$$

63. Given limit $= \lim_{x\to 0} \dfrac{(\sqrt{1+\sin x}-\sqrt{1-\sin x})}{x} \times \dfrac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}$

$= \lim_{x\to 0} \dfrac{(1+\sin x)-(1-\sin x)}{x(\sqrt{1+\sin x}+\sqrt{1-\sin x})}$

$= 2\times \lim_{x\to 0} \dfrac{\sin x}{x} \times \lim_{x\to 0} \dfrac{1}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})} = \left(2\times 1\times \dfrac{1}{2}\right) = 1.$

64. Given limit $= \lim_{x\to \frac{\pi}{6}} \dfrac{2-2\left(\dfrac{\sqrt{3}}{2}\cos x + \dfrac{1}{2}\sin x\right)}{(6x-\pi)^2}$

$= \lim_{x\to \frac{\pi}{6}} \dfrac{2\left[1-\left(\cos\dfrac{\pi}{6}\cos x + \sin\dfrac{\pi}{6}\sin x\right)\right]}{(6x-\pi)^2}$

$= \lim_{x\to \frac{\pi}{6}} \dfrac{2\left[1-\cos\left(x-\dfrac{\pi}{6}\right)\right]}{(6x-\pi)^2} = \lim_{\frac{x}{2}\to \frac{\pi}{12}} \dfrac{2\times 2\sin^2\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)}{144\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)^2}$

$= \dfrac{4}{144} \times \lim_{\frac{x}{2}\to \frac{\pi}{12}} \left\{\dfrac{\sin\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)}{\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)}\right\}^2 = \left(\dfrac{1}{36}\times 1^2\right) = \dfrac{1}{36}.$

65. Given limit $= \lim_{x\to 0} \dfrac{-2\sin\left(\dfrac{a+b}{2}\right)x \cdot \sin\left(\dfrac{a-b}{2}\right)x}{-2\sin^2\left(\dfrac{cx}{2}\right)}$

$= \lim_{x\to 0} \dfrac{\dfrac{\sin\left(\dfrac{a+b}{2}\right)x}{\left(\dfrac{a+b}{2}\right)x} \times \left(\dfrac{a+b}{2}\right)x \cdot \dfrac{\sin\left(\dfrac{a-b}{2}\right)x}{\left(\dfrac{a-b}{2}\right)x} \times \left(\dfrac{a-b}{2}\right)x}{\left\{\dfrac{\sin\left(\dfrac{cx}{2}\right)}{\left(\dfrac{cx}{2}\right)}\right\}^2}$

$= \left\{\left(\dfrac{a+b}{2}\right)\left(\dfrac{a-b}{2}\right)\times \dfrac{4}{c^2}\right\} \cdot \lim_{x\to 0} \dfrac{\sin\left(\dfrac{a+b}{2}\right)x}{\left(\dfrac{a+b}{2}\right)x} \cdot \lim_{x\to 0} \dfrac{\sin\left(\dfrac{a-b}{2}\right)x}{\left(\dfrac{a-b}{2}\right)x}$

$= \left\{\dfrac{(a^2-b^2)}{c^2}\times 1\times 1\right\} = \dfrac{(a^2-b^2)}{c^2}.$

66. Given limit $= \lim_{x\to a} \dfrac{\cos x - \cos a}{\cos x \sin a - \sin x \cos a} \cdot \sin x \sin a$

$$= \lim_{x \to a} \frac{-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{\sin(a-x)} \cdot \sin x \sin a$$

$$= \lim_{x \to 0} \frac{-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{-2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x-a}{2}\right)} \cdot \sin x \sin a$$

$$= \lim_{x \to 0} \frac{\sin\left(\frac{x+a}{2}\right)}{\cos\left(\frac{x-a}{2}\right)} \cdot \sin x \sin a = \sin^3 a.$$

67. Given limit $= \lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x + 1)(\tan x - 1)}{\cos\left(x + \frac{\pi}{4}\right)}$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x + 1)(\sin x - \cos x)}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)}$$

$$= -\lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x + 1)(\cos x - \sin x)}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)}$$

$$= -\sqrt{2} \lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x + 1)\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)}$$

$$= -\sqrt{2} \lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x + 1) \cdot \cos\left(x + \frac{\pi}{4}\right)}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)} = -\sqrt{2} \lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x + 1)}{\cos x}$$

$$= -\sqrt{2} \cdot \frac{1 \times (1+1)}{\left(\frac{1}{\sqrt{2}}\right)} = -4.$$

68. Given limit $= \lim_{x \to \frac{\pi}{2}} \frac{(\sqrt{2} - \sqrt{1+\sin x})}{\sqrt{2}\cos^2 x} \times \frac{(\sqrt{2} + \sqrt{1+\sin x})}{(\sqrt{2} + \sqrt{1+\sin x})}$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2 - (1+\sin x)}{\sqrt{2}(1-\sin^2 x)} \times \frac{1}{(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(1-\sin x)}{\sqrt{2}(1-\sin x)(1+\sin x)} \times \frac{1}{(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \frac{1}{\sqrt{2}} \cdot \lim_{x \to \frac{\pi}{2}} \frac{1}{(1+\sin x)(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{2 \times 2\sqrt{2}} = \frac{1}{8}.$$

69. Given limit $= \lim_{x\to\frac{\pi}{6}} \frac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 2} = \lim_{x\to\frac{\pi}{6}} (\operatorname{cosec} x + 2) = (2+2) = 4.$

70. Given limit $= \lim_{x\to\pi} \frac{(\sqrt{2+\cos x}-1)(\sqrt{2+\cos x}+1)}{(\pi - x)^2(\sqrt{2+\cos x}+1)}$

$= \lim_{x\to\pi} \frac{(1+\cos x)}{(\pi - x)^2(\sqrt{2+\cos x}+1)} = \lim_{h\to 0} \frac{1+\cos(\pi - h)}{h^2\{\sqrt{2+\cos(\pi - h)}+1\}}$

[putting $\pi - x = h$]

$= \lim_{h\to 0} \frac{(1-\cos h)}{h^2\{\sqrt{2-\cos h}+1\}} = \lim_{h\to 0} \frac{2\sin^2(h/2)}{\left(\frac{h}{2}\right)^2 \times 4 \times \{\sqrt{2-\cos h}+1\}}$

$= \frac{1}{2}\lim_{h\to 0} \left\{\frac{\sin(h/2)}{(h/2)}\right\}^2 \cdot \frac{1}{\lim_{h\to 0}\{\sqrt{2-\cos h}+1\}} = \left(\frac{1}{2}\times 1^2 \times \frac{1}{2}\right) = \frac{1}{4}.$

71. Given limit $= \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x \cdot \sqrt{2}\left\{\frac{1}{\sqrt{2}} - \sin x\right\}} = \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\left\{\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right\}}{\sqrt{2}\cos x\left(\sin\frac{\pi}{4} - \sin x\right)}$

$= \lim_{x\to\frac{\pi}{4}} \frac{\left\{\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x\right\}}{\cos x\left(\sin\frac{\pi}{4} - \sin x\right)}$

$= \lim_{x\to\frac{\pi}{4}} \frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos x \cdot 2\cos\left(\frac{\pi}{8}+\frac{x}{2}\right)\sin\left(\frac{\pi}{8}-\frac{x}{2}\right)}$

$= \lim_{x\to\frac{\pi}{4}} \frac{2\sin\left(\frac{\pi}{8}-\frac{x}{2}\right)\cos\left(\frac{\pi}{8}-\frac{x}{2}\right)}{\cos x \cdot 2\cos\left(\frac{\pi}{8}+\frac{x}{2}\right)\sin\left(\frac{\pi}{8}-\frac{x}{2}\right)} = \lim_{x\to\frac{\pi}{4}} \frac{\cos\left(\frac{\pi}{8}-\frac{x}{2}\right)}{\cos x \cdot \cos\left(\frac{\pi}{8}+\frac{x}{2}\right)}$

$= \frac{\cos\left(\frac{\pi}{8}-\frac{\pi}{8}\right)}{\cos\frac{\pi}{4}\cdot\cos\left(\frac{\pi}{8}+\frac{\pi}{8}\right)} = \frac{\cos 0}{\left(\cos\frac{\pi}{4}\right)^2} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2.$

72. Given limit $= \lim_{x\to\frac{\pi}{6}} \frac{2\sin^2 x + 2\sin x - \sin x - 1}{2\sin^2 x - 2\sin x - \sin x + 1}$

$= \lim_{x\to\frac{\pi}{6}} \frac{2\sin x(\sin x + 1) - (\sin x + 1)}{2\sin x(\sin x - 1) - (\sin x - 1)}$

$= \lim_{x\to\frac{\pi}{6}} \frac{(\sin x + 1)(2\sin x - 1)}{(\sin x - 1)(2\sin x - 1)} = \lim_{x\to\frac{\pi}{6}} \frac{(\sin x + 1)}{(\sin x - 1)}$

$$= \frac{\left(\sin\frac{\pi}{6}+1\right)}{\left(\sin\frac{\pi}{6}-1\right)} = \frac{\left(\frac{1}{2}+1\right)}{\left(\frac{1}{2}-1\right)} = \frac{3}{2}\times(-2) = -3.$$

LEFT-HAND LIMIT

Let $f(x)$ be a given function and let x approach to a from the left. Then, the limit of $f(x)$, as x approaches to a from the left, is denoted by $\lim\limits_{x\to a^-} f(x)$.

Method for Finding $\lim\limits_{x\to a^-} f(x)$

Replace x by $(a-h)$ and take the limit as $h\to 0$.

RIGHT-HAND LIMIT

Let $f(x)$ be a given function and let x approach to a from the right. Then, the limit of $f(x)$, as x approaches to a from the right, is denoted by $\lim\limits_{x\to a^+} f(x)$.

Method for Finding $\lim\limits_{x\to a^+} f(x)$

Replace x by $(a+h)$ and take the limit as $h\to 0$.

LIMIT OF A FUNCTION *We say that $\lim\limits_{x\to a} f(x)$ exists only when $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$ and this common value is known as $\lim\limits_{x\to a} f(x)$.*

SOLVED EXAMPLES

EXAMPLE 1 *Let* $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0. \end{cases}$

Find (i) $\lim\limits_{x\to 0} f(x)$ *(ii)* $\lim\limits_{x\to 1} f(x)$

SOLUTION (i) We have

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} 3(0+h+1) = \lim_{h\to 0} 3(h+1) = 3.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} 3(0-h+1) = \lim_{h\to 0} 3(-h+1) = 3.$$

$$\therefore \quad \lim_{x\to 0^+} (x) = \lim_{x\to 0^-} f(x) = 3.$$

Hence, $\lim\limits_{x\to 0} f(x) = 3.$

(ii) We have

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 3(1+h+1) = \lim_{h\to 0} 3(2+h) = 6.$$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 3(1-h+1) = \lim_{h\to 0} 3(2-h) = 6.$$

$$\therefore \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = 6.$$

Hence, $\lim_{x\to 1} f(x) = 6.$

EXAMPLE 2 *Let* $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1. \end{cases}$

Find $\lim_{x\to 1} f(x).$

SOLUTION We have

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \{-(1+h)^2 - 1\} = \lim_{h\to 0} (-2 - h^2 - 2h) = -2.$$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \{(1-h)^2 - 1\} = \lim_{h\to 0} (1 + h^2 - 2h - 1)$$

$$= \lim_{h\to 0} (h^2 - 2h) = 0.$$

$$\therefore \quad \lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x).$$

Hence, $\lim_{x\to 1} f(x)$ does not exist.

EXAMPLE 3 *Let* $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Find $\lim_{x\to 0} f(x).$

SOLUTION We have

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1.$$

Thus, $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x).$

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Caution Here note that $\frac{h}{h} = 1$, since $h \neq 0$.

EXAMPLE 4 *Let* $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1. \end{cases}$

If $\lim_{x\to 1} f(x) = f(1)$, *find the values of a and b.*

SOLUTION Clearly, $f(1) = 4.$

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \{b - a(1+h)\} = (b-a).$$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \{a + b(1-h)\} = (a+b).$$

Now, $\lim_{x\to 1} f(x) = f(1) = 4$

$\Rightarrow \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1) = 4$

$\Rightarrow \quad (b-a) = (a+b) = 4$

$\Rightarrow \quad b-a = 4 \text{ and } b+a = 4 \;\Rightarrow\; a = 0,\ b = 4.$

Hence, $a = 0$ and $b = 4$.

EXAMPLE 5 *Let* $f(x) = \begin{cases} |x|+1, & x < 0 \\ 0, & x = 0 \\ |x|-1, & x > 0 \end{cases}$

For what value(s) of a does $\lim_{x\to a} f(x)$ *exists?*

SOLUTION **Case 1.** *When* $a < 0$.

In this case, we have

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} \{|a+h|+1\} = |a|+1.$$

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} \{|a-h|+1\} = |a|+1.$$

$\therefore \quad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = |a|+1.$

So, in this case, $\lim_{x\to a} f(x)$ exists.

Case 2. *When* $a < 0$.

In this case, we have

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} \{|a+h|-1\} = |a|-1.$$

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} \{|a-h|-1\} = |a|-1.$$

$\therefore \quad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = |a|-1.$

So, in this case, $\lim_{x\to a} f(x)$ exists.

Case 3. *When* $a < 0$.

In this case, we have

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \{|h|-1\} = -1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \{|-h|+1\} = 1.$$

Thus, $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Thus, $\lim_{x\to a} f(x)$ exists only when $a \neq 0$.

EXAMPLE 6 *Let* $f(x) = \begin{cases} mx^2+n, & x < 0 \\ nx+m, & 0 \le x \le 1 \\ nx^3+m, & x > 1. \end{cases}$

For what values of integers m and n, $\lim_{x\to 0} f(x)$ *and* $\lim_{x\to 1} f(x)$ *both exist?*

SOLUTION We have

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} (nh+m) = m.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \{m(-h)^2 + n\}$$

$$= \lim_{h\to 0} (mh^2 + n) = n.$$

$\therefore$ $\lim_{x\to 0} f(x)$ exists only when $m = n$.

Now, $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \{n(1+h)^3 + m\}$

$$= \lim_{h\to 0} \{n(1+h^3+3h+3h^2) + m\} = (n+m).$$

And, $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \{n(1-h) + m\} = (n+m).$

$\therefore$ $\lim_{x\to 1} f(x) = (n+m).$

Hence, $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ both exist only when $m = n$.

EXAMPLE 7 *Let $a_1, a_2, \ldots, a_n$ be fixed real numbers and let*

$$f(x) = (x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n).$$

Find $\lim_{x\to a_1} f(x)$.

If $a \neq a_1, a_2, \ldots, a_n$, compute $\lim_{x\to a} f(x)$.

SOLUTION We have

$$\lim_{x\to a_1^+} f(x) = \lim_{h\to 0} f(a_1 + h)$$

$$= \lim_{h\to 0} \{(a_1+h-a_1)(a_1+h-a_2)(a_1+h-a_3)\ldots(a_1+h-a_n)\}$$

$$= \lim_{h\to 0} \{h(a_1+h-a_2)(a_1+h-a_3)\ldots(a_1+h-a_n)\}$$

$$= \{0\times(a_1-a_2)\times(a_1-a_3)\times\ldots\times(a_1-a_n) = 0.$$

$$\lim_{x\to a_1^-} f(x) = \lim_{h\to 0} f(a_1 - h)$$

$$= \lim_{h\to 0} \{(a_1-h-a_1)(a_1-h-a_2)(a_1-h-a_3)\ldots(a_1-h-a_n)\}$$

$$= \lim_{h\to 0} \{(-h)(a_1-h-a_2)(a_1-h-a_3)\ldots(a_1-h-a_n)\}$$

$$= \{0\times(a_1-a_2)\times(a_1-a_3)\times\ldots\times(a_1-a_n)\} = 0.$$

$\therefore$ $\lim_{x\to a_1^+} f(x) = \lim_{x\to a_1^-} f(x) = 0.$

Hence, $\lim_{x\to a_1} f(x) = 0.$

For any $a \neq a_1, a_2, \ldots, a_n$, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} \{(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)\}$$

$$= (a-a_1)(a-a_2)(a-a_3)\ldots(x-a_n).$$

EXAMPLE 8 *Let $f(x) = [x]$ = greatest integer less than or equal to x. For any integer k, show that* $\lim_{x\to k} f(x)$ *does not exist.*

SOLUTION We have

$$\lim_{x\to k^+} f(x) = \lim_{h\to 0} f(k+h) = \lim_{h\to 0} [k+h] = \lim_{h\to 0} k = k \quad \{\because\ [k+h] = k\}$$

and $\lim_{x\to k^-} f(x) = \lim_{h\to 0} f(k-h) = \lim_{h\to 0} [k-h] = \lim_{h\to 0} (k-1) = (k-1)$

$\{\because\ [k-h] = k-1\}$

Thus, $\lim_{x\to k^+} f(x) \neq \lim_{x\to k^-} f(x)$.

Hence, $\lim_{x\to k} f(x)$ does not exist.

EXAMPLE 9 *If f is an odd function and $\lim_{x\to 0} f(x)$ exists then prove that this limit must be 0.*

SOLUTION Let f be an odd function and let $\lim_{x\to 0} f(x)$ exist. Then

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$$

$$\Rightarrow \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h)$$

$$\Rightarrow \lim_{h\to 0} f(h) = \lim_{h\to 0} f(-h) = -\lim_{h\to 0} f(h) \quad [\because\ f \text{ being odd, } f(-h) = -f(h)]$$

$$\Rightarrow 2\lim_{h\to 0} f(h) = 0 \Rightarrow \lim_{h\to 0} f(h) = 0 \Rightarrow f(x) = 0.$$

Hence, $\lim_{x\to 0} f(x) = 0$.

EXAMPLE 10 *If f is an even function, prove that* $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$.

SOLUTION Let f be an even function. Then,

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} f(h) \qquad [\because\ f \text{ being even, } f(-h) = f(h)]$$

$$= \lim_{h\to 0} f(0+h) = \lim_{x\to 0^+} f(x).$$

Hence, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$.

EXAMPLE 11 *Show that* $\lim_{x\to 0}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$ *does not exist.*

SOLUTION Let $f(x) = \left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$. Then,

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0}\left(\frac{e^{1/h}-1}{e^{1/h}+1}\right) = \lim_{h\to 0}\left(\frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{1/h}}}\right) = \left(\frac{1-0}{1+0}\right) = 1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0}\left(\frac{e^{-1/h}-1}{e^{-1/h}+1}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1}\right) = \left(\frac{0-1}{0+1}\right) = -1.$$

Thus, $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

EXAMPLE 12 *Show that* $\lim_{x\to 0}\frac{x}{|x|}$ *does not exist.*

SOLUTION Let $f(x) = \frac{x}{|x|}$. Then,

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}\frac{h}{|h|} = \lim_{h\to 0}\frac{h}{h} = 1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}\frac{-h}{|-h|} = \lim_{h\to 0}\frac{-h}{h} = -1.$$

$\therefore \quad \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

EXERCISE 27C

1. If $f(x) = |x| - 3$, find $\lim_{x\to 3} f(x)$.

2. Let $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Show that $\lim_{x\to 0} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} \frac{|x-3|}{(x-3)}, & x \neq 3 \\ 0, & x = 3. \end{cases}$

Show that $\lim_{x\to 3} f(x)$ does not exist.

4. Let $f(x) = \begin{cases} 1+x^2, & 0 \leq x \leq 1 \\ 2-x, & x > 1. \end{cases}$

Show that $\lim_{x\to 1} f(x)$ does not exist.

5. Let $f(x) = \begin{cases} \frac{x-|x|}{x}, & x \neq 0 \\ 2, & x = 0. \end{cases}$

Show that $\lim_{x\to 0} f(x)$ does not exist.

6. Let $f(x)=\begin{cases}5x-4, & 0<x\le 1\\ 4x^3-3x, & 1<x<2.\end{cases}$

Find $\lim\limits_{x\to 1} f(x)$.

7. Let $f(x)=\begin{cases}4x-5, & x\le 2\\ x-a, & x>2.\end{cases}$

If $\lim\limits_{x\to 2} f(x)$ exists then find the value of a.

8. Let $f(x)=\begin{cases}\dfrac{3x}{|x|+2x}, & x\ne 0\\ 0, & x=0.\end{cases}$

Show that $\lim\limits_{x\to 0} f(x)$ does not exist.

9. Let $f(x)=\begin{cases}\cos x, & x\ge 0\\ x+k, & x<0.\end{cases}$

Find the value of k for which $\lim\limits_{x\to 0} f(x)$ exists.

10. Show that $\lim\limits_{x\to 0}\dfrac{1}{x}$ does not exist.

11. Show that $\lim\limits_{x\to 0}\dfrac{1}{|x|}=\infty$.

12. Show that $\lim\limits_{x\to 0} e^{-1/x}$ does not exist.

13. Show that $\lim\limits_{x\to 0}\sin\dfrac{1}{x}$ does not exist.

14. Show that $\lim\limits_{x\to 2}\dfrac{x}{[x]}$ does not exist.

15. Let $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & x\ne\dfrac{\pi}{2}\\ 3, & x=\dfrac{\pi}{2}.\end{cases}$

If $\lim\limits_{x\to\frac{\pi}{2}} f(x)=f\left(\dfrac{\pi}{2}\right)$, find the value of k.

ANSWERS (EXERCISE 27C)

1. 0 **6.** 1 **7.** $a=-1$ **9.** $k=1$ **10.** $k=6$

HINTS TO SOME SELECTED QUESTIONS

1. $\lim\limits_{x\to 3^+} f(x)=\lim\limits_{h\to 0} f(3+h)=\lim\limits_{h\to 0}\{|3+h|-3\}=\lim\limits_{h\to 0}(3+h-3)=\lim\limits_{h\to 0} h=0.$

$\lim\limits_{x\to 3^-} f(x)=\lim\limits_{h\to 0} f(3-h)=\lim\limits_{h\to 0}\{|3-h|-3\}=\lim\limits_{h\to 0}(3-h-3)=\lim\limits_{h\to 0}(-h)=0.$

$\therefore \quad \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = 0 \Rightarrow \lim_{x\to 3} f(x) = 0.$

2. $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{h}{|h|} = \lim_{h\to 0} \frac{h}{h} = 1$ $[\because\ h > 0]$

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{-h}{|-h|} = \lim_{h\to 0} \frac{-h}{h} = -1.$

$\therefore \quad \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$ and so $\lim_{x\to 0} f(x)$ does not exist.

3. $\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h) = \lim_{h\to 0} \frac{|3+h-3|}{(3+h-3)} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$ $[\because\ h > 0]$

$\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0} \frac{|3-h-3|}{(3-h-3)} = \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1.$

$\therefore \quad \lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$ and so $\lim_{x\to 3} f(x)$ does not exist.

4. $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \{2-(1+h)\} = \lim_{h\to 0} (1-h) = 1.$

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \{1+(1-h)^2\} = \lim_{h\to 0} (1+1+h^2-2h) = 2.$

$\therefore \quad \lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$ and so $\lim_{x\to 1} f(x)$ does not exist.

6. $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \{4(1+h)^3 - 3(1+h)\} = (4\times 1^3 - 3) = 1.$

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \{5(1-h) - 4\} = (5-4) = 1.$

$\therefore \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = 1$ and so $\lim_{x\to 1} f(x) = 1.$

7. $\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} (2+h-a) = (2-a).$

$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} \{4(2-h) - 5\} = (8-5) = 3.$

Since $\lim_{x\to 2} f(x)$ exists, we must have $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x).$

$\therefore \quad 2-a = 3 \Rightarrow a = (2-3) = -1.$

9. $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \cos h = 1.$

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \{(-h)+k\} = k.$

Since $\lim_{x\to 0} f(x)$ exists, we have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$ and hence $k = 1.$

10. Let $f(x) = \frac{1}{x}$. Then, $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{1}{h} = \infty.$

And, $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{1}{-h} = -\infty.$

$\therefore \quad \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$ and hence $\lim_{x\to 0} f(x)$ does not exist.

11. Let $f(x) = \frac{1}{|x|}$. Then,

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{1}{|h|} = \lim_{h\to 0} \frac{1}{h} = \infty$ $[\because\ h > 0]$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{1}{|-h|} = \lim_{h\to 0} \frac{1}{h} = \infty.$$

$\therefore \quad \lim_{x\to 0} \frac{1}{|x|} = \infty.$

12. Let $f(x) = e^{-1/x}$. Then,

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} e^{-1/h} = \lim_{h\to 0} \frac{1}{e^{1/h}} = \frac{1}{\infty} = 0.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} e^{1/h} = \infty. \qquad [\because\ e^{\infty} = \infty]$$

$\therefore \quad \lim_{x\to 0} f(x)$ does not exist.

13. Let $f(x) = \sin\frac{1}{x}$.

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \sin\frac{1}{h}.$$

Clearly, when h is given different values, then $\sin\frac{1}{h}$ oscillates between –1 and 1 and thus it does not approach to a definite value. So, $\lim_{h\to 0} \sin\frac{1}{h}$ does not exist.

Hence, $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

14. Let $f(x) = \frac{x}{[x]}$. Then,

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} \frac{2+h}{[2+h]} = 1 \qquad [\because\ h\to 0, [2+h] = 2]$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} \frac{2-h}{[2-h]} = \lim_{h\to 0} \frac{2-h}{1} = 2 \qquad [\because\ [2-h] = 1]$$

$\therefore \quad \lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$ and so $\lim_{x\to 2} f(x)$ does not exist.

15. $$\lim_{x\to \frac{\pi^+}{2}} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right) = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)} = \lim_{h\to 0} \frac{-k\sin h}{-2h}$$

$$= \lim_{h\to 0} \frac{k}{2}\left(\frac{\sin h}{h}\right) = \frac{k}{2}.$$

$\therefore \quad \frac{k}{2} = 3 \Rightarrow k = 6.$